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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

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CONTENTS.

Mathematical Papers, &c.

Solved Questions.

CORRIGENDA.

VOL. XIV.

p. 55, line 13, *for* $-4ab)\,s^2$ *read* $-2ab)\,s^2$.

VOL. XVI.

p. 22, line 8, *for* $(h-$ *read* $(h-x)$.
p. 22, line 15, *for* μ'^2 *read* μ^{12}.
p. 24, line 4 from bottom, *for* March...Times, *read* Reprint, vol. XV., p. 49.
p. 25, line 7, *for* $(x^2-1)^2$ *read* $(x^2-1)^{\frac{1}{2}}$.
p. 39, line 6 from bottom, *for* $\phi(\alpha\beta)$ *read* $\phi(\alpha, \beta)$.
p. 48, line 12, *dele* I., *and add* S. WATSON *and others*.
p. 64, line 23, *for* y *read* Y.
p. 67, line 19, *for* $\frac{4}{9}$ *read* $\frac{8}{9}$.
p. 74, line 13, *for* 2 *read* (2).
p. 79, line 13, *for* (9) *read* (10).
p. 81, line 6, *for* the 0 *read* the limits 0.
p. 108, line 11, *for* $s^2(x^2+y^2)$ *read* $s^2(x^2+y^2)^2$.

MATHEMATICS

FROM

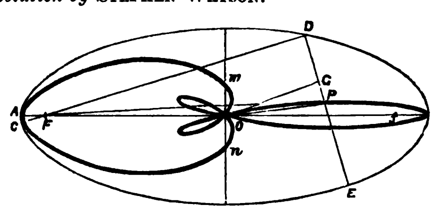
THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3383. (Proposed by the EDITOR.)—The vertex of a right angle moves round the curve of an ellipse, and one of its sides always passes through a focus; find the locus of the middle point of that part of the other side which lies within the ellipse: trace this curve, and show that its area is to that of the ellipse as $a^6-a^4b^2+15a^2b^4+b^6 : 2(a^2+b^2)^3$,
where a and b are the semiaxes of the ellipse.

I. *Solution by* STEPHEN WATSON.

Let O be the centre of the given ellipse; F, f the foci; CFD a focal chord; CDE a right angle meeting the ellipse in E; P the middle point of DE; and OG a perpendicular on DE. Join FP, OP; and put



$$\angle\,\mathrm{DFO}=\theta,\quad \mathrm{PFO}=\phi,\quad \mathrm{FD}=r=\frac{b^2}{a-c\cos\theta},\quad \mathrm{FP}=\rho.$$

Then, since OP is conjugate to the diameter parallel to DE,

$$\tan \mathrm{PO}f=\frac{b^2}{a^2}\tan\theta\,;$$

also
$$\mathrm{DG}=c\sin\theta,\quad \mathrm{OG}=r-c\cos\theta,$$

and
$$\mathrm{GP}=\mathrm{OG}\tan(\theta-\mathrm{PO}f)=\frac{c^2\tan\theta}{a^2+b^2\tan^2\theta}\,\mathrm{OG}\,;$$

hence $$\tan(\theta-\phi)=\frac{\mathrm{DG}+\mathrm{GP}}{r}=\frac{ac\sin\theta\sec^2\theta}{a^2+b^2\tan^2\theta},$$

therefore $$\sec^2(\theta-\phi)\,d(\theta-\phi)=\frac{ac\,(a^2+c^2\sin^2\theta)\sec^3\theta}{(a^2+b^2\tan^2\theta)^2}\,d\theta,$$

therefore $$\rho^2 d\phi=r^2\sec^2(\theta-\phi)\,d\phi$$

$$=r^2d\theta+r^2\tan^2(\theta-\phi)\,d\theta-\frac{acr^2\,(a^2+c^2\sin^2\theta)\sec^3\theta}{(a^2+b^2\tan^2\theta)^2}\,d\theta$$

$$=r^2d\theta+\frac{a^2c^2\tan^2\theta\sec^2\theta-ac\left\{a^2+(a^2+c^2)\tan^2\theta\right\}\sec\theta}{(a^2+b^2\tan^2\theta)^2}\,r^2d\theta\;......(1).$$

Putting in this for r its value above; then substituting $\pi-\phi$ for ϕ, in order to take in the case when the right angle is formed at C; adding the two results; integrating from $\theta=0$ to $\theta=\frac{1}{2}\pi$, and doubling, we have, for the area of the curve which is the locus of P,

$$\int\rho^2d\phi=\int_0^{\frac{1}{2}\pi}r^2d\theta+2b^4a^2c^2\int_0^{\infty}\frac{-2a^2-(3a^2+c^2)\,z^2-c^2z^4+a^2z^6}{(a^2+b^2z^2)^2\,(b^2+a^2z^2)^2}\,dz\;...(2),$$

where $z=\tan\theta$. Now $\int_0^{\frac{1}{2}\pi}r^2d\theta=\pi ab$, and the other part, being resolved into partial fractions and integrated in the usual way, gives

$$\frac{-a^6-7a^4b^2+9a^2b^4-b^6}{2\,(a^2+b^2)^3}\,(\pi ab);$$

hence (2) becomes $$\frac{a^6-a^4b^2+15a^2b^4+b^6}{2\,(a^2+b^2)^3}\,(\pi ab).$$

The polar equation of the curve in terms of ρ and θ is

$$\rho^2=\frac{b^4}{(a-c\cos\theta)^2}\left\{1+\frac{a^2c^2\tan^2\theta\sec^2\theta}{(a^2+b^2\tan^2\theta)^2}\right\},$$

and its general form (when $a>2b$) is that given in the diagram, touching the ellipse at A and B, and meeting the minor axis in m and n, so that

$$\mathrm{O}m=\mathrm{O}n=\frac{b^2}{a}=\tfrac{1}{2}\text{ parameter}.$$

The equation in x and y of the curve may be obtained as follows:—

The equations of FD, DE (D being denoted by x', y') are

$$y=\frac{y'}{x'+c}\,(x+c),\quad y-y'=-\frac{x'+c}{y'}\,(x-x')\;..............\;(1).$$

And since OP must coincide with the diameter conjugate to that parallel to DE, its equation is $$y=\frac{b^2y'}{a^2\,(x'+c)}\,x\;................................\;(2).$$

From (1) and (2) we have

$$x'=\frac{b^2x\,(a^2y^2+b^2x^2)-a^4cy^2}{a^4y^2+b^4x^2},\quad y'=\frac{a^2y\,(a^2y^2+b^2x^2+b^2cx)}{a^4y^2+b^4x^2}.$$

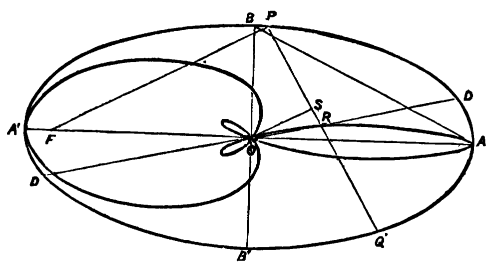
By substituting these in $\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, we shall have the equation of the curve; but this method does not seem to be so good as the one adopted above.

From (1) and (2) we also get

$$x=\frac{a^2(x'+c)\left\{y'^2+x'(x'+c)\right\}}{b^2y'^2+a^2(x'+c)^2},\quad y=\frac{b^2y'\left\{y'^2+x'(x'+c)\right\}}{b^2y'^2+a^2(x'+c)^2};$$

and from these the curve may be traced.

II. *Solution by the* PROPOSER.



1. Let FPQ be the right angle; R the middle point of the chord PQ; ABA′B′ the ellipse, O its centre, $OF=c=(a^2-b^2)^{\frac{1}{2}}$, $AB=s=(a^2+b^2)^{\frac{1}{2}}$, OS ($\perp$ PQ) $=\lambda$, OR $=r$, $\angle AOR=\theta$, $\angle AOS=OFP=\phi$; also let ψ be the eccentric angle of the point (D) where OR meets the ellipse, ρ the radius-vector of a point in the ellipse whose eccentric angle is ϕ, and ρ_1 the semi-diameter conjugate to ρ. Then we have

$$\rho^2=a^2\cos^2\phi+b^2\sin^2\phi=(a+c\sin\phi)(a-c\sin\phi),$$

$$\rho_1^2=a^2\sin^2\phi+b^2\cos^2\phi=(a+c\cos\phi)(a-c\cos\phi),$$

$$\lambda=FP-c\cos\phi=\frac{b^2}{a-c\cos\phi}-c\cos\phi=\frac{c^2\cos^2\phi-ac\cos\phi+b^2}{a-c\cos\phi};$$

therefore $\qquad \lambda\rho_1^2=ab^2-c^3\sin^2\phi\cos\phi.$

2. Now the equations of the chord PQ and the diameter DOD′ conjugate to it are $x\cos\phi+y\sin\phi=\lambda$, $a^2y\cos\phi-b^2x\sin\phi=0$ ...... (1, 2); and R is at the intersection of these two lines. Hence the equation of the locus of R is obtained by the elimination of ϕ from (1), (2); and it may be expressed in any one of the following forms:—

$$\frac{b^2(a^2+cx)(a^2y^2+b^2x^2)}{b^2cx+(a^2y^2+b^2x^2)}=a(a^4y^2+b^4x^2)^{\frac{1}{2}}\quad\text{.................. (3)},$$

$$\frac{a^2b^2c^2y^2(a^2y^2+b^2x^2)}{a^2b^2-(a^2y^2+b^2x^2)}=\left(\frac{a^4y^2+b^4x^2}{a^2+cx}\right)^2\quad\text{.................... (4)},$$

$$r=\frac{-a^4b^2c^3\sin^2\theta\cos\theta\pm ab^2(a^4\sin^2\theta+b^4\cos^2\theta)^{\frac{3}{2}}}{(a^2\sin^2\theta+b^2\cos^2\theta)(a^6\sin^2\theta+b^6\cos^2\theta)}\quad\text{........... (5)},$$

$$\frac{x}{a^2\cos\phi}=\frac{y}{b^2\sin\phi}=\frac{\lambda}{\rho^2}\quad\text{.......................... (6)}.$$

3. The most useful of these equations is (6), which gives x and y as functions of ϕ. The curve is symmetrical with respect to the axis of x, and it will take different forms according to the relative magnitudes of a, b. If $a > 2b$, the roots of the equation $\lambda = 0$, or $c^2 \cos^2 \phi - ac \cos \phi + b^2 = 0$, are real and unequal, and they give $\lambda = x = y = r = 0$. The origin is then a quadruple point, through which four real branches pass, and the curve is of the form shown in the figure.

If $a = 2b$, the roots of $\lambda = 0$ are real and equal, the small loops at O vanish, and two ceratoid cusps are formed there.

If $a < 2b$, the roots of $\lambda = 0$ are imaginary, and the curve does not pass through O, which is then a conjugate point.

The angles (θ_0) at which the several branches of the curve cut OA, may be found by putting $y = x \tan \theta$ in (3) or (4), and then ascertaining the values of θ_0 when $x = 0$; or by putting $r = 0$ in (5). We thus obtain

$$2a^3 \sec^2 \theta_0 = c^2 \left\{ 3a \pm (a^2 - 4b^2)^{\frac{1}{2}} \right\};$$

which, when $a = 2b$, gives $3 \sin \theta_0 = 1$, and $\theta_0 = 19° \, 28'$. From this again we see that θ_0 has four real and unequal values, two pairs of equal values, or four imaginary values, according as $a >$, $=$, or $< 2b$.

4. Now put Σ for the area of the locus of R; then, since

$$a^2 \tan \theta = b^2 \tan \phi = ab \tan \psi,$$

therefore $$\Sigma = \int \frac{1}{2} r^2 d\theta = \frac{1}{2} \int x^2 d (\tan \theta) = \frac{1}{2} \int x^2 d \left(\frac{b^2}{a^2} \tan \phi \right)$$

$$= \frac{b^2}{2a^2} \int x^2 \sec^2 \phi \, d\phi = \tfrac{1}{2} a^2 b^2 \int \left(\frac{\lambda^2}{\rho^4} \right) d\phi.$$

Hence the area may be found from any one of the following expressions:—

$$\Sigma = \int_0^{\pi} \frac{a^2 b^2}{\rho^4 \rho_1^4} \left\{ ab^2 - c^3 \sin^2 \phi \cos \phi \right\}^2 d\phi \quad \text{.............................. (7)},$$

$$\Sigma = ab \int_0^{\pi} \left\{ \frac{b (a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{3}{2}} - ac^3 \sin^2 \psi \cos \psi}{(a^4 \sin^2 \psi + b^4 \cos^2 \psi)} \right\}^2 d\psi \quad \text{...... (8)},$$

$$\Sigma = \tfrac{1}{2} a^2 b^2 \int \frac{(z^2 - az + b^2)^2}{(a - z)^2 (z^2 + b^2)^2} d\phi \quad \text{.................................... (9)},$$

z being put in (9) for $c \cos \phi$.

5. The integrals in (7) and (8) may be reduced, by the following properties of definite integrals (F being a functional symbol),

$$\int_0^{\pi} \mathrm{F} (\sin^2 \phi) \cos \phi \, d\phi = 0,$$

$$\int_0^{\pi} \mathrm{F} (\sin^2 \phi) \, d\phi = 2 \int_0^{\frac{1}{2}\pi} \mathrm{F} (\sin^2 \phi) \, d\phi = 2 \int_0^{\frac{1}{2}\pi} \mathrm{F} (\cos^2 \phi) \, d\phi.$$

Thus, from (7), we have

$$\Sigma = \int_0^{\frac{1}{2}\pi} \frac{2a^2 b^2}{\rho^4 \rho_1^4} (a^2 b^4 + c^6 \sin^4 \phi \cos^2 \phi) \, d\phi$$

$$= \int_0^{\frac{1}{2}\pi} \frac{2a^2 b^2}{\rho^4 \rho_1^4} (a^2 b^4 + c^6 \cos^4 \phi \sin^2 \phi) \, d\phi = \int_0^{\frac{1}{2}\pi} \frac{a^2 b^2}{\rho^4 \rho_1^4} (2a^2 b^4 + c^6 \sin^2 \phi \cos^2) \, d\phi$$

$$=\int_0^{\frac{1}{2}\pi}\frac{8a^2b^2c^2\,d\phi}{m^4-c^4\cos 4\phi}+\int_0^{\frac{1}{2}\pi}\frac{64a^4b^4(3b^2-a^2)\,d\phi}{(m^4-c^4\cos 4\phi)^2}$$

$$=\pi ab\left\{\frac{c^2}{s^2}+\frac{m^4}{2s^6}(3b^2-a^2)\right\}=\frac{\pi ab}{2s^6}(a^6-a^4b^2+15a^2b^4+b^6);$$

m^4 having been put for $2s^4-c^4$ or $a^4+6a^2b^2+b^4$.

Again, from (8), we have

$$\frac{s^6\Sigma}{ab}=s^6\int_0^{\pi}\frac{b^2(a^2\sin^2\psi+b^2\cos^2\psi)^3+a^2c^6\sin^4\psi\cos^2\psi}{(a^4\sin^2\psi+b^4\cos^2\psi)^2}\,d\psi$$

$$=\int_0^{\pi}(c^4s^2\cos^2\psi+a^4b^2+5a^2b^4)\,d\psi$$

$$-\int_0^{\pi}\frac{a^2b^4c^2(a^2+c^2)\,d\psi}{a^4\sin^2\psi+b^4\cos^2\psi}+\int_0^{\pi}\frac{2a^6b^8\,d\psi}{(a^4\sin^2\psi+b^4\cos^2\psi)^2}$$

$$=\tfrac{1}{2}\pi(a^6+a^4b^2+9a^2b^4+b^6)-\pi b^2c^2(a^2+c^2)+\pi b^2(a^4+b^4);$$

therefore $$\Sigma=\frac{\pi ab}{2s^6}(a^6-a^4b^2+15a^2b^4+b^6).$$

6. The preceding *definite integrals* give the *entire area* swept out by the radius-vector (r), and when $a>2b$ (as in the figure), the small loops at O are included twice. But resolving (9) into partial fractions, we readily obtain this *indefinite integral*

$$\frac{s^6\Sigma}{ab}=(3a^2b^4-a^4b^2)\left\{\tfrac{1}{2}\psi+\tan^{-1}\left(\frac{a+c}{b}\tan\tfrac{1}{2}\phi\right)\right\}+\tfrac{1}{2}s^6\psi+\tfrac{1}{8}c^4s^2\sin 2\psi$$

$$+\tfrac{1}{4}(3a^5b^3-ab^5)\log_e\left(\frac{a-c\sin\phi}{a+c\sin\phi}\right)+\tfrac{1}{2}ab^3cs^2\sin\phi\left(\frac{a+c\cos\phi}{\rho_1^2}-\frac{a}{\rho^2}\right);$$

and from this expression the area of any sector of the curve may be found. The double of this integral, between the limits $(\phi=\psi=0)$, $(\phi=\psi=\pi)$, gives $2s^6\Sigma=\pi ab(a^6-a^4b^2+15a^2b^4+b^6)$.

7. Hence, putting E $(=\pi ab)$ for the area of the ellipse, we have

$$\frac{\Sigma}{E}=\frac{a^6-a^4b^2+15a^2b^4+b^6}{2a^6+6a^4b^2+6a^2b^4+2b^6}\qquad(10).$$

As $a:b$ approaches the limit unity (or c the limit 0), both the ellipse and the locus of R approach the circle as their limit; and when $a=b$, (10) gives $\Sigma=E=\pi a^2$. If $a=2b$, then (10) gives $\Sigma:E=109:250$.

8. The following is a more general form of the problem :—

A straight line moves with its ends on two given curves, and is always inclined at a given angle to the line joining one of its ends with a fixed point; find the locus of the point which divides the moving line into two parts which have to one another a given ratio.

Putting (x,y) for the dividing point, and (h,k), (m,n) for the ends of the moving line, we can form five equations; two to express the conditions that the ends of the moving line are in the given curves, two that (x,y) divides the moving line in a given ratio, and a fifth the constant inclination. The locus of (x,y) may be found by eliminating h,k,m,n from these five equations.

9. For example, in the given Question, we should have

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}=\frac{m^2}{a^2}+\frac{n^2}{b^2}=1 \quad\dots\dots\dots\dots\dots\dots\dots\dots (11),$$

$$\frac{h}{x}+\frac{m}{x}=\frac{k}{y}+\frac{n}{y}=2 \quad\dots\dots\dots\dots\dots\dots\dots\dots (12),$$

$$\frac{k-n}{h-m}+\frac{h+c}{k}=0 \quad\dots\dots\dots\dots\dots\dots\dots\dots (13);$$

(11) expressing that P and Q are in the ellipse, (12) that R is the middle of PQ, and (13) that PQ is perpendicular to PF.

From (11) and (12) we obtain

$$\frac{x}{a^2}(h-m)+\frac{y}{b^2}(k-n)=0=\frac{x}{a^2}(h- \quad +\frac{y}{b^2}(k-y),$$

therefore $$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2} \quad\dots\dots\dots\dots\dots\dots\dots\dots (14).$$

From (13) and (14) we have $\dfrac{h+c}{k}=\dfrac{b^2x}{a^2y}$ .... (15),

which is the equation of the diameter DOD′ conjugate to PQ, and corresponds to (2).

Putting $\nu^4=a^2y^2+b^2x^2$, $\mu^6=a^4y^2+b^4x^2$, we find, from (14) and (15),

$$\mu^6h=\nu^4b^2x-a^4cy^2,\quad \mu^6k=\nu^4a^2y+a^2b^2cxy \quad\dots\dots\dots\dots (16).$$

From (11) and (16) we have $a^2b^2\mu'^2=\nu^4\left\{\mu'^2+a^2b^2c^2y^2(a^2+cx)^2\right\}$,

which is identical with (4).

3407. (Proposed by Rev. W. A. Whitworth, M.A.)—A heavy particle is suspended from a fixed point by a string which it can statically stretch to double its natural length. If the particle is projected so as to describe a horizontal circle with uniform velocity, prove that the natural length of the string is half the harmonic mean between the height and the slant side of the cone which the string traces out.

I. *Solution by* J. Hopkinson, D.Sc., B.A.

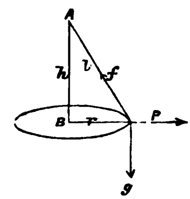
Let a be the natural length of the string, b the height of the cone described, and c its slant side. Let mass of particle be taken as unit of mass.

Tension in string $= g\dfrac{c-a}{a}$. Resolve vertically

$$g=g\frac{c-a}{a}\cdot\frac{b}{c},\quad\text{or}\quad \frac{1}{b}+\frac{1}{c}=\frac{1}{a},$$

which proves the proposition.

II. *Solution by* G. S. CARR.



Let P be the particle, &c.; then ABP is a triangle of forces or accelerations $g,\ f,\ \dfrac{v^2}{r}$; therefore

$$\frac{l}{f}=\frac{h}{g}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1).$$

Also since (mg) stretches the string of natural length a to the length $2a$; therefore (mf) stretches it to the length $\quad a+\dfrac{f}{g}a=l;$

therefore $\quad l=a\dfrac{g+f}{g}=a\dfrac{h+l}{h}$ by (1);

therefore $\quad a=\dfrac{hl}{h+l}.$

3429. (Proposed by J. W. L. GLAISHER, B.A.)—Prove that

$$\int_{-\infty}^{\infty} e^{-\left(x^4+\frac{a^4}{x^4}\right)+4(3a)^{\frac{1}{2}}\left(x^3-\frac{a^3}{x^3}\right)-14a\left(x^2+\frac{a^2}{x^2}\right)}\,dx=\tfrac{1}{2}\Gamma\left(\tfrac{1}{4}\right)e^{21a^2}.$$

Solution by the PROPOSER.

The result to be proved may be written

$$\int_{-\infty}^{\infty} e^{-\left(x-\frac{a}{x}-\sqrt{3a}\right)^4}dx=\Gamma\left(\tfrac{1}{4}\right),$$

and by Boole's Theorem $\displaystyle\int_{-\infty}^{\infty} f\left(x-\frac{a}{x}\right)dx=\int_{-\infty}^{\infty} f(x)\,dx$

(*Philosophical Transactions*, 1858, p. 780), the left-hand side is equal to

$$\int_{-\infty}^{\infty} e^{-[x-(3a)^{\frac{1}{2}}]^4}dx=\int_{-\infty}^{\infty} e^{-x^4}dx$$

$$=2\int_0^{\infty} e^{-x^4}dx=\tfrac{1}{2}\int_0^{\infty} e^{-v}v^{-\frac{3}{4}}dv=\tfrac{1}{2}\Gamma\left(\tfrac{1}{4}\right).$$

3281. (Proposed by the Rev. R. TOWNSEND, M.A., F.R.S.)—Show that, for a homogeneous solid parallelepiped of any form and dimensions, the three principal axes at the centre of gravity coincide in direction with

those of the solid inscribed ellipsoid which touches at the six centres of gravity of its six faces; and that, for each of the three coincident axes, and therefore for every axis passing through their common centre of gravity, the moment of inertia of the parallelepiped is to that of the ellipsoid in the same constant ratio, viz., that of 10 to π.

Solution by the PROPOSER.

For, the three lines AA′, BB′, CC′, connecting their three pairs of opposite contacts A and A′, B and B′, C and C′, and passing through their common centre of gravity O, forming, for both solids alike, a common oblique system of coordinate axes, for which, manifestly,

$$\Sigma\,(y'z'dm) = 0, \quad \Sigma\,(z'x'dm) = 0, \quad \Sigma\,(x'y'dm) = 0,$$

and for which, as may be seen without difficulty,

$$\Sigma\,(x'^2dm) : \Sigma\,(y'^2dm) : \Sigma\,(z'^2dm) = (\mathrm{AA}')^2 : (\mathrm{BB}')^2 : (\mathrm{CC}')^2,$$

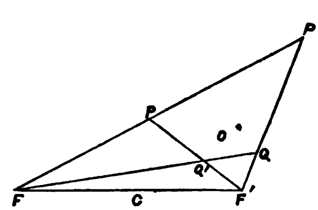
therefore, &c., as regards the first part; the unique system of rectangular axes for which $\Sigma\,(yzdm) = 0$, $\Sigma\,(zxdm) = 0$, $\Sigma\,(xydm) = 0$ being consequently given by the same system of equations for both.

Denoting by m_1 and m_2 the masses of the parallelepiped and ellipsoid respectively, and by I_1 and I_2 their moments of inertia round any common axis passing through O; then since, for the three coincident principal planes through O determined for both alike by the aforesaid system of equations,

$$\Sigma\,(x_1^2dm_1) : \Sigma\,(x_2^2dm_2) = \Sigma\,(y_1^2dm_1) : \Sigma\,(y_2^2dm_2) = \Sigma\,(z_1^2dm_1) : \Sigma\,(z_2^2dm_2) = 5m_1 : 3m_2,$$

and since $m_1 : m_2 = 6 : \pi$, therefore $\mathrm{I}_1 : \mathrm{I}_2 = 10 : \pi$; and therefore, &c., as regards the second part.

NOTE ON NEWTON'S THEOREM. *By* C. TAYLOR, M.A.

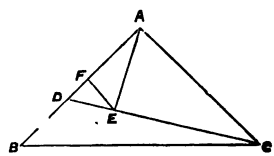


Let O be the centre of a circle inscribed in a quadrilateral whose diagonals are PQ, P′Q′, FF′. The sum or difference of one pair of opposite sides is equal to the sum or difference of the other pair (see *Messenger of Mathematics*, Vol. III., p. 201). Hence it may be shown indirectly that the three diagonals are bisected by one diameter of the circle. For since, as the figure is drawn, FP + PF′ = FQ + QF′, an ellipse with foci F, F′ passes through P, Q; and its tangents at P, Q are the bisectors of the angles F′PP′, FQP′, that is, OP, OQ. Now in the ellipse (centre C), CO bisects the chord of contact PQ. Similarly, in the confocal hyperbola through P′, Q′, the same line CO bisects the chord of contact P′Q′.

For the general case of an inscribed conic, Dr. BOOTH has given a proof by his method of tangential coordinates in the March number of the *Educational Times*. For an indirect solution, see an article entitled *Mechanical Solutions of Geometrical Problems*, in Vol. VIII., p. 127, of the *Quarterly Journal of Pure and Applied Mathematics*.

3432. (Proposed by Miss GREAVES.)—The triangle ABC is right-angled at A, and has AB = AC = 1. In AB a point D is taken such that AD is a mean proportional between BD and DC; find the lengths of AD, DB, DC.

Solution by the PROPOSER; S. C. W. GRANT; R. TUCKER, M.A.; *and others.*



Let CD = x, then AD = $(x^2-1)^{\frac12}$, and we have $x\left\{1-(x^2-1)^{\frac12}\right\} = x^2-1$, an equation which reduces to

$$x^3-x = \tfrac12.$$

This equation being resolved by HORNER's method, we find

$$\text{CD} = 1{\cdot}1914871, \quad \text{AD} = {\cdot}6477998, \quad \text{DB} = {\cdot}3522002.$$

[If we put the angle ACD = θ, we shall have

$$\sec\theta\,(1-\tan\theta) = \tan^2\theta, \quad \text{or} \quad \cos\theta - \sin\theta = \sin^2\theta,$$

or $$\sin^2\theta + \sin\theta = \cos\theta, \quad \text{or} \quad \sin^3\theta + \sin^2\theta + \sin\theta = 1.$$

From the third of these equations we see that if AE, EF be perpendicular to CD, DA respectively, then CE = AE + EF.]

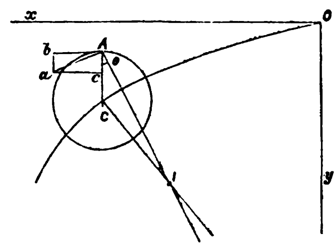
2621. (Proposed by the Rev. M. M. U. WILKINSON, M.A.)—If four points be taken at random on the surface of a sphere, show that the chance of their all lying on some one hemisphere is $\frac78$.

Solution by STEPHEN WATSON.

If great circles of the sphere be described through each pair of three of the points, they will divide the surface of the sphere into eight portions, in *seven* of which the fourth point may lie so that the four points may all be in some one hemisphere. Moreover, when the three points take all positions on the surface of the sphere, the eight portions will all pass through the same magnitudes; hence the required chance is $\frac78$.

3093. (Proposed by G. M. MINCHIN, B.A.)—A sphere is made to rotate about a diameter with a given angular velocity, and is then projected horizontally with a given velocity, in a plane perpendicular to the axis of rotation. Show that the locus of the instantaneous axis is a parabolic cylinder.

Solution by the PROPOSER.



The centre of gravity C of the sphere describes a parabola, and the body rotates uniformly about C. Let ω = the angular velocity about C; β = the horizontal velocity of projection. Consider the highest point A in the body. We shall find the direction of A's absolute motion thus:—

A rotates about C through a space $= a\omega$ (a being the radius of the sphere) and has a horizontal velocity $= \beta$, as also a vertical velocity $= gt$. Hence, drawing $Ab = a\omega + \beta$, and $Ac = gt$, the diagonal Aa is the direction of A's absolute motion. Draw the normal to the parabola at C, and a perpendicular to Aa at A. These lines will intersect in I, the instantaneous axis. To find the locus of I, eliminate t from the equations of AI and CI. These equations are easily seen to be

$$\frac{\beta t - x}{y + a - \frac{1}{2}gt^2} = \frac{gt}{a\omega + \beta} \quad \left(\text{since } \tan\theta = \frac{ab}{bA} = \frac{gt}{a\omega + \beta}\right) \quad \ldots\ldots\ldots\ldots(1),$$

$$(y - \tfrac{1}{2}gt^2) + \frac{\beta}{g}(x - \beta t) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The latter gives $$y - \tfrac{1}{2}gt^2 = \frac{\beta(\beta t - x)}{gt},$$

which, when substituted in (1), gives $t = \dfrac{\omega x}{\omega\beta - g}$,

and this, again, substituted in (2), gives the parabola

$$y - \frac{g\omega^2 x^2}{2(g - \omega\beta)^2} - \frac{\beta}{\omega} = 0, \quad \text{or} \quad x^2 = \frac{2(g - \omega\beta)^2}{g\omega^2}\left(y - \frac{\beta}{\omega}\right).$$

3416. (Proposed by the Rev. A. F. TORRY, M.A.)—A magnifying glass consists of two convex lenses, whose thickness, as also the distance between them, may be neglected. When used by a person who can see most distinctly at a distance of eight inches, the ratios of the magnifying powers of the first lens alone, the second alone, and the two combined, are 3 : 4 : 5. Find the focal lengths of the lenses used.

I. *Solution by* T. MITCHESON.

From the given data we obviously get

$$\frac{f+8}{f} : \frac{f_1+8}{f_1} : \frac{F+8}{F} = 3 : 4 : 5,$$

where f, f_1 are the focal lengths of the lenses, and F that of the combination. But $\frac{1}{F} = \frac{1}{f} + \frac{1}{f}$; substituting the equivalent of F, and eliminating f_1, we get $f = 16$, whence $f_1 = 8$.

II. *Solution by the* Rev. R. TOWNSEND, M.A., F.R.S.

Denoting by x, y, z the three distances from the three lenses of the objects whose images are formed at the common distance d (= 8 inches) of most distinct vision of the observer, and by u and v the two focal lengths required; since then, by well-known elementary formulæ,

$$\frac{1}{x}-\frac{1}{d}=\frac{1}{u},\quad \frac{1}{y}-\frac{1}{d}=\frac{1}{v},\quad \frac{1}{z}-\frac{1}{d}=\frac{1}{u}+\frac{1}{v};$$

and since, by hypothesis,

$$\frac{d}{x}:\frac{d}{y}:\frac{d}{z}=a:b:c\,(=3:4:5),$$

therefore, at once, we have

$$u=\frac{a+b-c}{c-b}\,d=2d=16 \text{ inches},\qquad v=\frac{a+b-c}{c-a}\,d=d=8 \text{ inches};$$

and therefore, &c.

2990. (Proposed by the EDITOR.)—Find three square *integers* in arithmetical progression, such that the square root of each increased by unity shall be a rational square.

I. *Solution by* ASHER B. EVANS, M.A.

Represent the three square integers in arithmetical progression by x^2, $25x^2$, $49x^2$, and let $x+1=(r+1)^2$; then the three conditions become

$$x+1=(r+1)^2 \text{ ....(1)},\quad 5x+1=5r^2+10r+1=\square \text{ ....(2)},$$

$$7x+1=7r^2+14r+1=\square \text{ ....(3)}.$$

Assume $(nr+1)$ for the root of (3); then $r=\dfrac{14-2n}{n^2-7}$, and (2) becomes, after being multiplied by $(n^2-7)^2$,

$$(n^2-7)^2+20\,(7-n)\,(n^2-7)+20\,(7-n)^2=\square \text{ .... (4)}.$$

Assume $(n^2-7)-6\,(7-n)$ for the root of (4); then we have

$$n=3,\quad r=\frac{14-2n}{n^2-7}=4,\quad x=(r+1)^2-1=24.$$

Hence 24^2, 120^2, 168^2 are the numbers.

II. *Solution by* ARTEMAS MARTIN.

Let x^2, $25x^2$, and $49x^2$ be the numbers. Then must

$$x+1=\square,\quad 5x+1=\square,\ \text{and}\ 7x+1=\square \text{ .... (1, 2, 3)}.$$

Put $x+1=a^2$, and we have $x=a^2-1$.

Substituting in (2) and (3), these equations become

$$5a^2-4=\square,\quad 7a^2-6=\square \text{ .... (4, 5)}.$$

Both of these expressions are satisfied by $a = 1$; let, therefore, $a = b+1$, and we have $\quad 5b^2+10b+1 = \square, \quad 7b^2+14b+1 = \square$ .... (6, 7).

Put (6) $= (1+nb)^2$, and we get $b = \dfrac{10-2n}{n^2-5}$.

Substituting in (7), and reducing, the equation becomes

$$n^4-21n^3+158n^2-140n+25 = \square = (n^2+2n+5)^2, \text{ suppose.}$$

By involution and reduction we obtain $2n^2-9n = -10$;

whence $\quad n = 2\frac{1}{2}$ or 2, $\quad \therefore\ b = 4$ or -6, $\quad a = \pm 5$, $\quad x = 24$,

and the numbers are 24^2, 120^2, 168^2.

3265. (Proposed by R. Tucker, M.A.)—The escribed radius (to side a) of a triangle is equal to the circumscribed radius, find what relation subsists between the cosines of the angles of the triangle; and if also the mean escribed radius (to side b) is a geometric mean between the two other escribed radii, find the ratio of a to c.

Solution by the Proposer.

From the given relation we have

$$\frac{a\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A} = \frac{a}{2\sin A}, \quad\text{or}\quad 4\sin\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C = 1;$$

whence $\quad \cos B + \cos C = \cos A$ .... (1).

From the second condition we have, since $r_1 = s\tan\frac{1}{2}A$,

$$\tan^2\tfrac{1}{2}B = \tan\tfrac{1}{2}A\tan\tfrac{1}{2}C, \quad\text{or}\quad (s-b)^2 = (s-a)(s-c),$$

reducing, we get $\quad b = \dfrac{a^2+{}^2}{a+c}$ .... (2).

From (1), $\quad b(a^2+c^2-b^2) + c(a^2+b^2-c^2) = a(b^2+c^2-a^2)$.

Substituting from (2), we have

$$2ac(a^2+c^2)^2 + 2ac(a+c)(a^3+ac^2+a^2c-c^3) = 2ac(a+c)(a^2c+ac^2-a^3+c^3);$$

that is, $\quad a^4+2a^2c^2+c^4+a^4+2a^2c^2+2a^3c-c^4 = 2a^2c^2-a^4+2ac^3+c^4$,

or, $\quad 3a^4+2a^2c^2+2a^3c-2ac^3-c^4 = 0$,

hence the ratio required (m) will be found from

$$3m^4+2m^3+2m^2-2m-1 = 0.$$

Multiplying the roots by 3, the equation becomes

$$x^4+2x^3+6x^2-18x-27 = 0,$$

and of this there is a positive root between 2 and 3, which will be found to be 2·10521, whence the required ratio $m = \cdot70173$.

3339. (Proposed by J. J. WALKER, M.A.)—If l, m, n are the arcs drawn from the angles ABC of a spherical triangle to the middle points of the opposite sides, and λ, μ, ν their segments adjacent to the angles from which they are drawn; prove that

$$\frac{\tan l}{\tan \lambda} = \frac{1+\cos b+\cos c}{\cos b+\cos c}, \quad \frac{\tan m}{\tan \mu} = \text{\&c.} \ldots\ldots \text{\&c.} \ldots\ldots\ldots (1),$$

$$\tan^2 l = \frac{\sin^2 b+\sin^2 c+2\sin b\sin c\cos A}{(\cos b+\cos c)^2} \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Solution by the PROPOSER.

1. Calling the other segments of the median arcs λ', μ', ν' respectively, and considering the triangle whose sides are $l, c, \frac{1}{2}a$ as cut by the transversal arc n,
$$\frac{\sin\lambda'}{\sin\lambda} = \frac{\sin\frac{1}{2}a}{\sin a} = \frac{1}{2\cos\frac{1}{2}a} = \frac{\cos l}{\cos b+\cos c},$$
whence
$$\frac{\sin(l-\lambda)}{\sin\lambda\cos l} \text{ or } \frac{\tan l}{\tan\lambda} - 1 = \frac{1}{\cos b+\cos c}, \quad \frac{\tan l}{\tan\lambda} = \frac{1+\cos b+\cos c}{\cos b+\cos c}.$$

2. $$\cos^2 l = \frac{(\cos b+\cos c)^2}{4\cos^2\frac{1}{2}a} = \frac{(\cos b+\cos c)^2}{2(1+\cos a)},$$

therefore $$\sin^2 l = \frac{2(1+\cos a)-(\cos^2 b+\cos^2 c+2\cos b\cos c)}{2(1+\cos a)}$$
$$= \frac{\sin^2 b+\sin^2 c+2(\cos a-\cos b\cos c)}{2(1+\cos a)} = \frac{\sin^2 b+\sin^2 c+2\sin b\sin c\cos A}{2(1+\cos a)},$$

whence $$\tan^2 l = \frac{\sin^2 l}{\cos^2 l} = \frac{\sin^2 b+\sin^2 c+2\sin b\sin c\cos A}{(\cos b+\cos c)^2}.$$

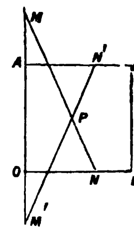
PROBABILITY NOTATION. No. 2. *By* HUGH MCCOLL.

Definitions:—The symbol $p(r)$ denotes the probability of the occurrence of the rth event (specified in a given table), and $p(:r)$ denotes the probability of its non-occurrence; so that $p(r)+p(:r)=1$. The symbol $p(m_a . n_{:c} : r_e)$ denotes the probability of the occurrence of the mth and nth events, and the non-occurrence of the rth, the lower numbers a, $:c$ and e (which are merely parenthetical, and not absolutely necessary) serving to remind us that the occurrence of the mth and nth events, and the non-occurrence of the rth, imply the occurrence of the ath and eth, and the non-occurrence of the cth. The expression $a > b = c > d$ asserts that $a > b$ and $c > d$ are equivalent conditions, each implying the other.

This notation I shall employ, for convenience, in the solution of the following problem (Question 3385), and then assume the result in my solution of Mr. WATSON's Question 3440:—

A point is taken at random in a given rectangle. What is the probability that a straight line drawn at random through this point will cut two given opposite sides?

Let AOBC be the rectangle; AC and OB the given opposite sides; P the random point; x, y the coordinates of P; and M, N the points at which the random line meets the lines $x=0$, $y=0$ respectively. Let $OB=1$, $OA=a$, $OM=m$, $ON=n$; and let θ denote the angle PNS, where S is the point whose coordinates are $y=0$, $x=-\infty$. Let Q denote the probability that $m>a$ and $1>n$; all values of x between 0 and 1, of y between 0 and a, and of θ between 0 and π, being equally probable. Then it is evident, *à priori*, that the required probability is 2Q, since the same variations occur when the line takes such a position as M′PN′ in the diagram.



No. of Event.	Table of Reference.
1	$y > y_1$, in which $y_1 = a - x\tan\theta$
2	$y_2 > y$, in which $y_2 = (1-x)\tan\theta$
3	$y_2 > y_1 = \frac{1}{2}\pi > \theta > \tan^{-1} a$
4	$0 > y_1 = x > x_1$, in which $x_1 = a\cot\theta$
5	$y_2 > a = x_2 > x$, in which $x_2 = 1 - x_1$
6	$x_2 > x_1 = \frac{1}{2}\pi > \theta > \tan^{-1}(2a)$
7	$\tan^{-1}(2a) > \theta > \tan^{-1} a$ implies $: 6$

Now, since $m = y + x\tan\theta$, and $n = x + y\cot\theta$, we get at once

$$Q = p(1.2) = p(1.2.3), \text{ for } 1.2 \text{ implies } 3,$$
$$= p(4_1.2.3) + p(:4.1.2.3) = Q_1 + Q_2 \text{ say;}$$

$$Q_1 = p(4_1.2.3) = p(4_1.5_2.3) + p(4_1:5.2.3)$$
$$= p(4_1.5_2.6_3) + p(4_1:5.2.3), \text{ for } 4_1.5_2 \text{ implies } 6_3,$$
$$= p(4_1.5_2.6_3) + p(4_1.2.7_3) + p(6_3.2:5),$$
$$\text{for } 4_1.7_3 \text{ implies } :5.3, \text{ and } 6_3:5 \text{ implies } 4_1.3,$$
$$= P_1 + P_2 + P_3 \text{ say;}$$

$$Q_2 = p(:4.1.2.3) = p(:4.1.5_2.3) + p(:4.1:5.2.3)$$
$$= p(:4.1.6_3) + p(7_3.1.5_2) + p(:4.1:5.2.3),$$
$$\text{for } :4.6_3 \text{ implies } 5_2.3, \text{ and } 7_3.5_2 \text{ implies } :4.3,$$
$$= p(:4.1.6_3) + p(7_3.1.5_2) + p(:4.1:5.2.7_3),$$
$$\text{for } :4:5.3 \text{ implies } 7_3,$$
$$= P_4 + P_5 + P_6 \text{ say;}$$

hence $$Q = P_1 + P_2 + P_3 + P_4 + P_5 + P_6.$$

The problem has now been resolved into six elementary cases, which require nothing further than a straightforward application of the Integral Calculus, the limits, in each case, being obtained at once from the accompanying table. For P_1 we have the integral $\frac{1}{\pi}\iint d\theta\, dx$, the limits being $\frac{1}{2}\pi > \theta > \tan^{-1}(2a)$ and $x_2 > x > x_1$; and for each of the five others

we have the integral $\frac{1}{\pi a}\iiint d\theta\, dx\, dy$; the limits for P_2 being $\tan^{-1}(2a) > \theta > \tan^{-1} a$, $1 > x > x_1$, $y_2 > y > 0$; for P_3 the limits $\frac{1}{2}\pi > \theta > \tan^{-1}(2a)$, $x_2 > x > 0$, $y_2 > y > 0$; for P_4 the limits $\frac{1}{2}\pi > \theta > \tan^{-1}(2a)$, $x_1 > x > 0$, $a > y > y_1$; for P_5 the limits $\tan^{-1}(2a) > \theta > \tan^{-1} a$, $x_2 > x > 0$, $a > y > y_1$; and for P_6 the limits $\tan^{-1}(2a) > \theta > \tan^{-1} a$, $x_1 > x > x_2$, $y_2 > y > y_1$. We thus get finally

$$2Q = \frac{1}{\pi}\left\{2\cot^{-1} a - a\log\left(\frac{a^2+1}{a^2}\right)\right\} = \phi(a) \text{ say.}$$

The probability that the random line cuts the opposite sides OA and BC is evidently $\phi(a^{-1})$. To obtain this result, we have only to bear in mind that $a = \text{OA} + \text{OB}$, and $a^{-1} = \text{OB} \div \text{OA}$.

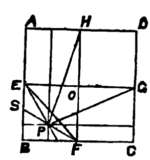
The chance of the random line cutting two given adjacent sides is

$$\frac{1}{4}\left\{1 - \phi(a) - \phi(a^{-1})\right\} = \frac{1}{4\pi a}(a^2+1)\log(a^2+1) - \frac{a}{2\pi}\log a$$
$$= \phi_1(a) \text{ say} = \phi_1(a^{-1}).$$

[Other Solutions of Quest. 3386 may be seen on pp. 83—86 of Vol. XV. of the *Reprint*.]

3440. (Proposed by S. Watson.)—A line is drawn at random across a window containing four equal rectangular panes; what are the respective chances that it crosses one, two, or three of the panes?

Solution by Hugh McColl.



Let the window be represented by the accompanying figure. A point being taken at random in the whole rectangle AC, and a random line drawn through it, we are required to find the respective chances of this random line crossing one, two, or three of the four rectangles AO, EF, OC, HG. Let p_1, p_2, p_3 denote the required chances. Let $\text{EB} \div \text{BF} = a$. If the random point be restricted to one rectangle, say EF, the chance of the line crossing its outer adjacent sides EB, BF is $\phi_1(a)$, in which $\phi_1(a)$ has the value assigned to it in the preceding article; and the chance of the random point falling in this rectangle is $\frac{1}{4}$. The chance of the random line crossing this one rectangle only is therefore $\frac{1}{4}\phi_1(a)$; and the chance of its crossing *some* one rectangle only is $\phi_1(a)$. Hence $p_1 = \phi_1(a)$.

Again, since the chance of the line crossing AB and BF is $\frac{1}{2}\phi_1(2a)$, and that of its crossing EB and BF is $\frac{1}{4}\phi_1(a)$, the chance of its crossing AE and BF must be $\frac{1}{2}\phi_1(2a) - \frac{1}{4}\phi_1(a)$; the chance of its crossing AH and EB is also $\frac{1}{2}\phi_1(2a) - \frac{1}{4}\phi_1(a)$; and the chance of its crossing AH and BF is $\frac{1}{2}\phi(2a)$, for the value of which see the preceding article. The chance therefore of the random line crossing the two rectangles AO and EF only is $\frac{1}{2}\phi(2a) + \phi_1(2a) - \frac{1}{2}\phi_1(a)$ = the chance of its crossing the two rectangles HG and OC only. In like manner, we find that the chance of the line

crossing the rectangles EF and OC only = the chance of its crossing AO and HG only $= \frac{1}{4}\phi\left(\frac{2}{a}\right)+\phi_1\left(\frac{2}{a}\right)-\frac{1}{2}\phi_1(a)$. Hence we have

$$p_2 = \phi(2a)+\phi\left(\frac{2}{a}\right)+2\phi_1(2a)+2\phi_1\left(\frac{2}{a}\right)-2\phi_1(a); \quad \text{and} \quad p_3 = 1-p_1-p_2.$$

It is evident, *à priori*, that the values of p_1, p_2, p_3 will not be altered by changing a into a^{-1}. The following results are, I believe, correct, so far as the approximations are carried:—

When $a = 1$, $p_1 = \cdot 11032$, $p_2 = \cdot 48379$, $p_3 = \cdot 40589$;
when $a = \frac{4}{3}$ or $\frac{3}{4}$, $p_1 = \cdot 10833$, $p_2 = \cdot 49251$, $p_3 = \cdot 39916$;
when $a = 2$ or $\frac{1}{2}$, $p_1 = \cdot 09955$, $p_2 = \cdot 53100$, $p_3 = \cdot 36945$.

[We may obtain, by direct integration, another solution, which we add here in order to verify Mr. McColl's results.

Taking a random line to mean a line drawn at random through a point taken at random on the window, it is evident that all the different cases (πa in all) will be obtained by supposing the line to revolve through two right angles for every position of the point P on the rectangle EBFO. Then, drawing the additional lines in the diagram, we see at once that the line will cross *one* pane only if it fall within the angle EPS; and, in order to do so, P must fall somewhere on the triangle EBF. The line will cross *three* squares if it fall within the angle HPG, and also if it cuts the two sides OE, OF. Hence, estimating x, y along BF, BE for p_1, and along OE, OF for p_3, we have

$$p_1 = \frac{1}{\pi a}\int_0^1\int_0^{a(1-x)} dx\,dy\left\{\tan^{-1}\left(\frac{a-y}{x}\right)-\tan^{-1}\left(\frac{y}{1-x}\right)\right\}$$

$$= \frac{1}{4\pi a}(a^2+1)\log(a^2+1)-\frac{a}{2\pi}\log a;$$

$$p_3 = \frac{1}{\pi a}\int_0^1\int_0^a dx\,dy\left\{\tan^{-1}\left(\frac{a+y}{x}\right)-\tan^{-1}\left(\frac{y}{1+x}\right)\right\}+p_1$$

$$= \frac{1}{4\pi a}\left\{(a^2+1)\log(a^2+1)+(4a^2-1)\log(4a^2+1)-(a^2-4)\log(a^2+4)\right\}$$
$$+\frac{2}{\pi}\left\{\cot^{-1}\frac{2}{3}\left(a+\frac{1}{a}\right)-\left(a+\frac{1}{a}\right)\log 2-a\log a\right\};$$

and when $a=1$, these probabilities become

$$p_1 = \frac{1}{2\pi}\log 2, \quad p_3 = \frac{1}{2\pi}\log\frac{125}{128}+\frac{2}{\pi}\cot^{-1}\frac{4}{3};$$

which agree, both in the general form and in the particular case, with Mr. McColl's developed results.]

3441. (Proposed by J. F. Moulton, M.A.)—If l, m, n be direction-cosines, show that all conicoids represented by

$$x^2+y^2+z^2+2lyz+2mzx+2nxy = 1$$

will be similar that have the product lmn the same.

I. *Solution by* MILLICENT COLQUHOUN.

The equation to find the axes is $\begin{vmatrix} 1-h, & n\ , & m \\ n\ , & 1-h, & l \\ m\ , & l\ , & 1-h \end{vmatrix} = 0,$

or $$h^3 - 3h^2 + 2h - 2lmn = 0.$$

The roots of this are in an invariable ratio if lmn is constant, and only then; therefore that lmn is constant is a sufficient and necessary condition of similarity; but it should also be noticed that if lmn is constant, the conics are identical, but differently placed.

II. *Solution by* G. S. CARR.

Transforming the axes so that the products may vanish, the discriminating cubic becomes, in this case,

$$h^3 - 3h^2 + (3 - l^2 - m^2 - n^2)\,h + (l^2 + m^2 + n^2 - 2lmn - 1) = 0.$$

But if l, m, n are direction cosines, $l^2 + m^2 + n^2 = 1$; and if lmn is constant, the coefficients of this cubic, and therefore the roots, are the same for all values of l, m, n. But these roots are the coefficients α, β, γ in the transformed equation $$\alpha x^2 + \beta y^2 + \gamma z^2 = 1;$$

and as the absolute term is not altered by the transformation, the resulting equation is always the same, and therefore the proposed equation represents not only similar but *equal* conicoids, the variations in l, m, n affecting nothing beyond the position of the principal axes.

3307. (Proposed by the Rev. R. TOWNSEND, M.A., F.R.S.)—If OX, OY, OZ be any system of three axes, rectangular or oblique, at the centre of gravity O of any system of masses, for which $\Sigma\,(myz) = 0$, $\Sigma\,(mzx) = 0$, $\Sigma\,(mxy) = 0$; show that the entire aggregate mass M of the system may be divided into any four parts M_a, M_b, M_c, M_d, and concentrated, respectively, half of M_a at each of the two distances $\pm a$ on OX for which $M_a a^2 = \Sigma\,(mx^2)$, half of M_b at each of the two $\pm b$ on OY for which $M_b b^2 = \Sigma\,(my^2)$, half of M_c at each of the two $\pm c$ on OZ for which $M_c c^2 = \Sigma\,(mz^2)$, and the whole of M_d at O itself, without altering the moment of inertia of the system with respect to any plane passing through O, and therefore with respect to any plane whatever, or the product of inertia of the system with respect to any two planes passing through O, and therefore with respect to any two planes whatever.

Solution by the PROPOSER.

For, if p and p' be the perpendicular distances of any point xyz from any two planes passing through O, and $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ their direction angles with respect to OX, OY, OZ respectively; then, since

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma, \quad \text{and} \quad p' = x\cos\alpha' + y\cos\beta' + z\cos\gamma',$$

therefore, by virtue of the given relations,

$$\Sigma(mp^2) = \Sigma(mx^2)\cos^2\alpha + \Sigma(my^2)\cos^2\beta + \Sigma(mz^2)\cos^2\gamma$$
$$= M_a\,.\,a^2\cos^2\alpha + M_b\,.\,b^2\cos^2\beta + M_c\,.\,c^2\cos^2\gamma$$
$$= \tfrac{1}{2}M_a\,.\,2a^2\cos^2\alpha + \tfrac{1}{2}M_b\,.\,2b^2\cos^2\beta + \tfrac{1}{2}M_c\,.\,2c^2\cos^2\gamma,$$

and $\Sigma(mpp') = \Sigma(mx^2)\cos\alpha\cos\alpha' + \Sigma(my^2)\cos\beta\cos\beta' + \Sigma(mz^2)\cos\gamma\cos\gamma'$

$$= M_a\,.\,a^2\cos\alpha\cos\alpha' + M_b\,.\,b^2\cos\beta\cos\beta' + M_c\,.\,c^2\cos\gamma\cos\gamma'$$
$$= \tfrac{1}{2}M_a\,.\,2a\cos\alpha\,.\,a\cos\alpha' + \tfrac{1}{2}M_b\,.\,2b\cos\beta\,.\,b\cos\beta' + \tfrac{1}{2}M_c\,.\,2c\cos\gamma\,.\,c\cos\gamma';$$

and therefore, &c. as regards any plane or planes passing through O; and since $M_a + M_b + M_c + M_d = M$, therefore, &c. as regards any plane or planes whatever.

3410. (Proposed by C. TAYLOR, M.A.)—A parallelogram being inscribed in a rectangular hyperbola, show that a chord which subtends equal angles at the extremities of a side, subtends equal angles at the extremities of the opposite side.

Solution by the REV. R. TOWNSEND, M.A., F.R.S.

This is a particular case of the more general property that, if a chord PQ of any conic subtend equal angles at the extremities A and B of another chord AB, it does so at the extremities C and D of every parallel chord CD. Which is evident from the consideration that, if E and F be the intersections of PQ with AB and CD respectively, then, since always $EP.EQ : EA.EB = FP.FQ : FC.FD$, therefore, if the four points P, Q, A, B be concyclic, so are also the four P, Q, C, D, and therefore, &c.

3222. (Proposed by A. B. EVANS, M.A.)—Find the least integral value of x that will satisfy each of the conditions

$$940751x^2 + 1 = \square, \quad 940751x^2 + 38 = \square.$$

Solution by the PROPOSER; A. MARTIN; *and others.*

Let $940751x^2 + 1 = y^2$, and $940751x^2 + 38 = z^2$; then

$$y^2 - 940751x^2 = 1, \quad z^2 - 940751x^2 = 38 \quad\ldots\ldots\ldots\ldots\ (1, 2).$$

By calculating the partial quotients arising from extracting the square root of 940751 to the end of the first period, we find this period to commence with the second partial quotient and to contain ninety-six terms. The first partial quotient is 969; the ninety-six partial quotients in the first period are arranged in the following table:—

Rank	0	1	2	3	4	5	6	7	8	9
0	969	1	12	50	1	34	3	2	4	1
10	16	1	51	2	15	1	1	6	2	1
20	19	8	1	34	2	1	1	1	2	3
30	1	8	3	1	76	1	5	7	1	1
40	3	1	1	3	6	1	3	1	38	1
50	3	1	6	3	1	1	3	1	1	7
60	5	1	76	1	3	8	1	3	2	1
70	1	1	2	34	1	8	19	1	2	6
80	1	1	15	2	51	1	16	1	4	2
90	3	34	1	50	12	1	1938			

Since the rank of 1938, the last partial quotient in the first period, is even, the numerator and denominator of the converging fraction corresponding to this partial quotient will be the least integral values of y and x that will satisfy equation (1). By calculating this converging fraction we find, after an immense amount of labour,

$y = 1052442265723679403769386042332565332655403940191478220799$,

$x = 108507794585987643465094782581372488576176266730010272 0$.

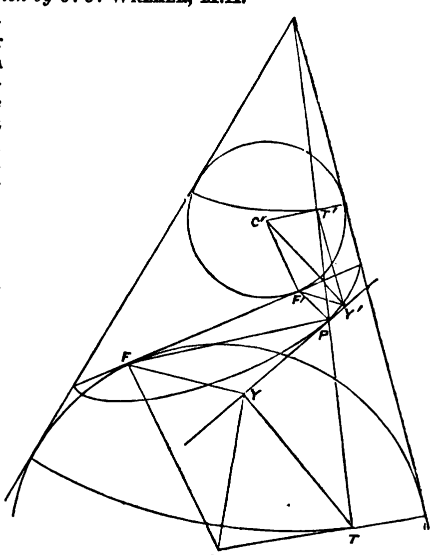
The *denominators* of the *complete quotients* arising from extracting the square root of 940751 by the method of continued fractions to the end of the first period are exhibited in the following table:—

Rank	0	1	2	3	4	5	6	7	8	9
0	1	1790	149	38	1849	55	557	766	361	1510
10	109	1798	37	931	121	1006	925	283	625	1279
20	98	215	1682	55	730	931	821	950	673	454
30	1369	215	473	1454	25	1606	317	251	1010	899
40	473	950	925	523	274	1375	401	1510	49	1510
50	401	1375	274	523	925	950	473	899	1010	251
60	317	1606	25	1454	473	215	1369	454	673	950
70	821	931	730	55	1682	215	98	1279	625	283
80	925	1006	121	931	37	1798	109	1510	361	766
90	557	55	1849	38	149	1790	1			

Since 38 is not found among the denominators of complete quotients of an *even* rank in the first period, equation (2) is impossible in integers; and therefore, the condition $940751x^2 + 38 = \square$ is impossible while x is an integer.

3453. (Proposed by C. TAYLOR, M.A.)—In the right circular cone, if FY, F′Y′ be the focal perpendiculars upon the tangent at any point (P) of a section, and C, C′ the centres of the inscribed spheres which touch the plane of section, show that CY, YY′, C′Y′ are at right angles to one another.

Solution by J. J. WALKER, M.A.



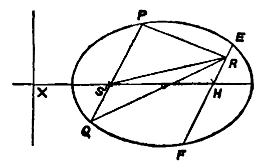
It was originally, I believe, shown in a paper which will be found in an early volume of the *Transactions of the Cambridge Philosophical Society* that the spheres C, C′ will touch the plane of the section at its foci F, F′. (In a paper which will be found in the *Cambridge and Dublin Mathematical Journal* for Feb. 1852, I gave a generalization of the theorem for the oblique cone.)

Suppose the side of the cone which passes through P to touch the spheres at the points T, T′. Since YY′ is perpendicular both to FY and FC, it is also perpendicular to CY. Similarly it is perpendicular to C′Y′. In the two triangles TPY, FPY, PT and PF are equal, being tangents to the sphere C; and similarly YT = YF, while the base PY is common; consequently ∠PYT = ∠FYP = a right angle, and therefore TY, CY, FY are in the same plane; also ∠YPT = ∠YPF, and ∠YTP = ∠YFP.

Similarly, it may be shown that T′Y′, C′Y′, F′Y′ lie in one plane, perpendicular to YY′, and therefore parallel to the plane CFY; also that the ∠Y′PT′ = ∠Y′PF′, and ∠Y′T′P = ∠YF′P. Hence it follows that, since the angles YPT, Y′PT′ are vertically opposite, ∠YPF = ∠Y′PF′ (not to assume this well-known property), and therefore that ∠YFP = ∠Y′F′P, and the ∠YTP = ∠Y′T′P; consequently the lines YT, Y′T′ are parallel. Also YF, Y′F′ are parallel, and YT, Y′T′ are drawn on opposite sides of the plane of YF, Y′F′, whence it follows that the angles FYT, F′Y′T′ are supplementary, and that CY, C′Y′, which are the bisectors of these angles respectively, are perpendicular one to the other.

3214. (Proposed by R. TUCKER, M.A.)—Two parallel focal chords of an ellipse are drawn, and a point is taken on one such that the focal vectors of the other subtend equal angles at it, prove that this point lies on a cubic through the foci and the foot of the further directrix, and that the rectangle under the vertical vectors of the cubic is constant.

Solution by the PROPOSER.



Let R be the point on EF a focal chord parallel to the focal chord PQ, such that $\angle$ PRS = $\angle$ QRS; then if $\angle$ PSH $= \theta$, SP $= r$, SQ $= r'$, HR $= \rho$, we have

$$PR : QR = r : r' \text{............ (1)},$$

$$PR^2 = 4a^2e^2 + (r-\rho)^2 - 2(r-\rho)\, 2ae \cos\theta,$$

$$RQ^2 = 4a^2e^2 + (r'+\rho)^2 + 2(r'+\rho)\, 2ae \cos\theta.$$

From (1) we get

$$(r-r')\left\{4a^2e^2+\rho^2+4ae\rho\cos\theta\right\} + 2rr'\left\{\rho + 2ae\cos\theta\right\} = 0 \text{...... (2)};$$

but $\qquad r = \dfrac{c}{1-e\cos\theta}$, therefore $\dfrac{r-r'}{2rr'} = \dfrac{e\cos\theta}{c}$,

hence (2) becomes $\rho^2 e\cos\theta + a\rho\left\{4e^2\cos^2\theta + 1 - e^2\right\} + 2a^2e\cos\theta\,(1+e^2) = 0$;

or changing to rectangular axes, we have

$$Y^2\left\{eX + a(1-e^2)\right\} + eX^3 + aX^2(1+3e^2) + 2a^2e(1+e^2)X = 0,$$

or $\qquad Y^2\left\{eX + a(1-e^2)\right\} + X(X+2ae)\left\{eX + a(1+e^2)\right\} = 0,$

which shows that the curve is a cubic symmetrical with regard to the axis of x, that it passes through the foci and the foot of the directrix furthest from H, and has for an asymptote a line parallel to the directrix at the same distance from H as the nearer directrix.

The polar equation shows that the rectangle under the vertical vectors is constant.

3283. (Proposed by the Rev. G. H. HOPKINS, M.A.)—If the transverse section of a right cylinder be an ellipse with axes $2a$, $2b$, determine the surface upon which will lie the foci of all the sections made by planes passing through a fixed point in the axis of the cylinder.

Solution by Professor WOLSTENHOLME; Rev. J. L. KITCHIN, M.A.; *and others.*

If the point be taken as origin, the equation of the cylinder may be taken $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Let $z = px + qy$ be the equation of any plane through

the point, then the foci of the section are the points of intersection of tangents drawn through the circular points at infinity in this plane, which are given by $x^2+y^2+z^2=0$, $z=px+qy$; hence at these points $x^2+y^2+(px+qy)^2=0$. Let $x=ky$ be a solution of this equation, then $x-ky=r$ will touch $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $r^2=a^2+b^2k^2$, and the corresponding equations are the projections of the tangent lines through the circular points. We have, then, at any point on such a line,

$$(x-ky)^2=a^2+b^2k^2, \quad \text{or} \quad k^2(b^2-y^2)+2kxy+a^2-x^2=0,$$

where k is given by the equation $k^2+1+(kp+q)^2=0$.

Hence, for the real points on these lines, we have

$$\frac{b^2-y^2}{1+p^2}=\frac{xy}{pq}=\frac{a^2-x^2}{1+q^2},$$

which, combined with $z=px+qy$, are the equations determining the foci; and eliminating p and q, we have the equation of the surface locus, which is a circular sextic.

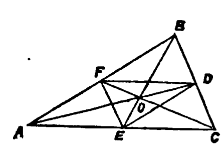
2830. (Proposed by R. Tucker, M.A.)—Straight lines are drawn from the angles of a triangle through a point O within it to meet the opposite sides; denoting the triangles formed by joining the points of section by α, β, γ, find the locus of O when

$$l\alpha+m\beta+n\gamma=\text{a constant}, \quad \alpha\gamma=k\beta^2, \quad \frac{l}{\alpha}+\frac{m}{\beta}+\frac{n}{\gamma}=\text{a constant}\ldots(1,2,3).$$

$$l\frac{\alpha}{\beta}+m\frac{\beta}{\gamma}+n\frac{\gamma}{\alpha}=\text{constant}, \quad l\alpha\beta+m\beta\gamma+n\gamma\alpha=\text{constant}\ldots\ldots(4,5).$$

Solution by the Proposer.

Take AC, AB for axes, then, corresponding to the concurrent point O, or (x, y), we have



$$D\left(\frac{bcx}{by+cx}, \frac{bcy}{by+cx}\right), \quad E\left(\frac{cx}{c-y}, 0\right), \quad F\left(0, \frac{by}{b-x}\right).$$

Therefore $\alpha=\frac{1}{2}\frac{bcxy}{(b-x)(c-y)}\sin A$, $\beta=\frac{1}{2}\frac{bcx(by+cx-bc)}{(b-x)(by+cx)}\sin A$,

$$\gamma=\frac{1}{2}\frac{bcy(by+cx-bc)}{(c-y)(by+cx)}\sin A.$$

Hence substituting in (1), we have

$$lxy(by+cx)+\left\{mx(c-y)+ny(b-x)\right\}(by+cx-bc)$$
$$=k(b-x)(c-y)(by+cx),$$

a cubic which always passes through the vertices of the triangle, and also passes through the *centroid* if $l-m-n=4k$.

The equation can also be written in the form

$$(by+cx)\left\{(l-m-n-k)\,xy+cx\,(m+k)+by\,(n+k)-kbc\right\}$$
$$= bc\left\{mx\,(c-y)+ny\,(b-x)\right\},$$

showing that the cubics pass through the point

$$x = b\,\frac{n-m}{n+m},\quad y = -c\,\frac{n-m}{n+m}.$$

Passing on to (2), we have the quartic curve

$$y^2\,(b-x)\,(by+cx) = kx\,(c-y)^2\,(by+cx-bc),$$

which clearly passes through the points A, C and B′, B″, the opposite angular points to A of parallelograms with AB, AC (or CA produced an equal distance to the left of A) as adjacent sides.

Equation (3) takes the form

$$l\,(b-x)\,(c-y)\,(by+cx-bc)+my\,(b-x)\,(by+cx)$$
$$+nx\,(c-y)\,(by+cx) = kxy\,(by+cx-bc),$$

which shows that the locus of O is a cubic curve.

Substituting in (4), and reducing, we get

$$lxy^2\,(ly+cx)^2\,(b-x)+mx^2\,(c-y)^2\,(by+cx)\,(by+cx-bc)$$
$$+ny\,(by+cx-bc)^2\,(b-x)^2\,(c-y) = kxy\,(b-x)\,(c-y)\,(by+cx)\,(by+cx-bc),$$

a sextic curve passing through A, B, C, and other points readily determinable.

Also in the case of (5) we have another sextic, whose equation is

$$lx^2y\,(c-y)\,(by+cx)\,(by+cx-bc)+mxy\,(b-x)\,(c-y)\,(by+cx-bc)^2$$
$$+nxy^2\,(b-x)\,(by+cx)\,(by+cx-bc) = k\,(b-x)^2\,(c-y)^2\,(by+cx)^2,$$

passing through A, B, C and other fixed points. As most of the peculiarities of the curves are readily discovered, I have merely given the equations as above.

3360. (Proposed by S. Roberts, M.A.)—If a curve be traced on the card of a mariner's compass, and the compass be moved without oscillation, with its centre on a circle, what is the nature of the envelope of the traced curve in its successive positions?

Solution by the Proposer.

The envelope in question is a parallel of the curve.

A parallel of the curve $\phi\,(x,y)=0$ is the envelope of the circles

$$(x-\alpha)^2+(y-\beta)^2 = r^2,\quad \phi\,(\alpha\beta)=0.$$

But writing α' for $x-\alpha$, β' for $y-\beta$, we get the equivalent system of curves

$$\phi\,(x-\alpha',\,y-\beta')=0,\quad \alpha'^2+\beta'^2=r^2.$$

The above conclusion follows from this immediately, for the travelling curve is translated without rotation in consequence of the polarity of the needle to which the card of a mariner's compass is fixed.

3417. (Proposed by W. HOGG.)—A centre of force C moves along the straight line OA with a uniform velocity, attracting, with a force varying directly as the first power of the distance, a particle P which is moving in the same straight line; having given the initial position of C, and both the initial position and the initial velocity of P, find the position of P at any time.

I. *Solution by* G. S. CARR.

Let v = velocity of centre C; then, for an attractive force,

$$\frac{d^2x}{dt^2} = -\mu(x - vt) \text{ .... (1)}; \quad \text{therefore } \frac{d^4x}{dt^4} = -\mu\frac{d^2x}{dt^2}.$$

Let $\frac{d^2x}{dt^2} = y$, then $\frac{d^2y}{dt^2} = -\mu y$;

therefore by two integrations (TAIT and STEELE'S *Dynamics*, Art. 83) we obtain $y = \mathrm{A}\cos(\mu^{\frac{1}{2}}t + \mathrm{B})$.

Equating this with (1), we have

$$x = vt - \frac{\mathrm{A}\cos(\mu^{\frac{1}{2}}t + \mathrm{B})}{\mu} \text{ ......(2)}; \quad \therefore \frac{dx}{dt} = v + \frac{\mathrm{A}\sin(\mu^{\frac{1}{2}}t + \mathrm{B})}{\mu^{\frac{1}{2}}} \text{......(3)}.$$

Putting $t = 0$ in (2) and (3), we obtain the initial distance of P from C and the initial velocity of P, viz.,

$$a = -\frac{\mathrm{A}\cos\mathrm{B}}{\mu}, \quad \text{and} \quad \mathrm{V} = v + \frac{\mathrm{A}\sin\mathrm{B}}{\mu^{\frac{1}{2}}},$$

which furnish A and B. Then (2) gives the position, and (3) the velocity, of P at any time t.

II. *Solution by the* Rev. R. TOWNSEND, M.A., F.R.S.

Denoting by a and b the initial distances of P and C from any fixed origin O on the line of motion OA, by x and y their distances from the same at any time t, by u and v their initial velocities, and by μ the absolute force of attraction; then since, by hypothesis, $\frac{d^2x}{dt^2} = \mu(y - x)$, and since by same $y = vt + b$, where v is the constant velocity of C; therefore $\frac{d^2x}{dt^2} + \mu x = \mu(vt + b)$, and therefore $x = vt + b + h\cos\mu^{\frac{1}{2}}t + k\sin\mu^{\frac{1}{2}}t$, the two constants h and k, as given by the initial values a and u of x and $\frac{dx}{dt}$, having for values $(a - b)$ and $\left(\frac{u - v}{\mu^{\frac{1}{2}}}\right)$ respectively; and therefore, &c.

III. *Solution by the* PROPOSER.

Let a, a' be the initial distances of C, P from O,

β = the uniform velocity of C, β' = the initial velocity of P,
x = the distance of P from O at time t, μ = the absolute force of attraction.

Then $\frac{d^2x}{dt^2} = -\mu(x-a-\beta t)$, or $\frac{dx^2}{dt^2}(x-a-\beta t)+\mu(x-a-\beta t) = 0$.

Assume $$x = \mathrm{A}\cos(\mu^{\frac{1}{2}}t)+\mathrm{B}\sin(\mu^{\frac{1}{2}}t)+a+\beta t \quad\text{.................. (1)},$$

therefore $$\frac{dx}{dt} = -\mathrm{A}\mu^{\frac{1}{2}}\sin(\mu^{\frac{1}{2}}t)+\mathrm{B}\mu^{\frac{1}{2}}\cos(\mu^{\frac{1}{2}}t)+\beta \quad\text{.............. (2)}.$$

But $\mathrm{A} = a'-a$ in the beginning, and

$$\beta' = \mathrm{B}\mu^{\frac{1}{2}}+\beta;\ \text{therefore}\ \mathrm{B} = \frac{\beta'-\beta}{\mu^{\frac{1}{2}}};$$

therefore $$x = (a'-a)\cos(\mu^{\frac{1}{2}}t)+\frac{\beta'-\beta}{\mu^{\frac{1}{2}}}\sin(\mu^{\frac{1}{2}}t)+a+\beta t$$

$$= a+\beta t+\frac{\beta'-\beta}{\mu^{\frac{1}{2}}}\sin(\mu^{\frac{1}{2}}t)+(a'-a)\cos(\mu^{\frac{1}{2}}t).$$

Supposing the conditions the same as before, only the force repulsive; then

$$\frac{d^2x}{dt^2} = \mu(x-a-\beta t),\ \text{or}\ \frac{dx^2}{dt^2}(x-a-\beta t)-\mu(x-a-\beta t) = 0.$$

Assume $$x = \mathrm{AE}^{\mu^{\frac{1}{2}}t}+\mathrm{BE}^{-\mu^{\frac{1}{2}}t}+a+\beta t \quad\text{..................... (1)},$$

therefore $$\frac{dx}{dt} = \mathrm{A}\mu^{\frac{1}{2}}\mathrm{E}^{\mu^{\frac{1}{2}}t}-\mathrm{B}\mu^{\frac{1}{2}}\mathrm{E}^{-\mu^{\frac{1}{2}}t}+\beta \quad\text{..................... (2)}.$$

But in the beginning $x = a'$, $t = 0$, and velocity $= \beta'$; taking these circumstances in equations (1) and (2), we have

$$a' = \mathrm{A}+\mathrm{B}+a,\quad \beta' = \mathrm{A}\mu^{\frac{1}{2}}-\mathrm{B}\mu^{\frac{1}{2}}+\beta;$$

therefore $$\mathrm{A} = \frac{a'-a}{2}-\frac{\beta-\beta'}{2\mu^{\frac{1}{2}}},\quad \mathrm{B} = \frac{a'-a}{2}+\frac{\beta-\beta'}{2\mu^{\frac{1}{2}}}.$$

Substituting these values in equation (1), we have

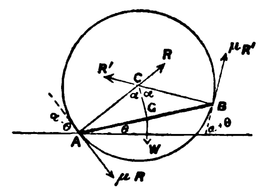
$$x = \left(\frac{a'-a}{2}-\frac{\beta-\beta'}{\beta}\right)\mathrm{E}^{\mu^{\frac{1}{2}}t}+\left(\frac{a'-a}{2}+\frac{\beta-\beta'}{2}\right)\mathrm{E}^{-\mu^{\frac{1}{2}}t}+a+\beta t$$

$$= a+\beta t-\left\{\mu^{\frac{1}{2}}(a-a')+(\beta-\beta')\right\}\frac{\mathrm{E}^{\mu^{\frac{1}{2}}t}}{2\mu^{\frac{1}{2}}}-\left\{\mu^{\frac{1}{2}}(a-a')-(\beta-\beta')\right\}\frac{\mathrm{E}^{\mu^{\frac{1}{2}}t}}{2\mu^{\frac{1}{2}}}.$$

3396. (Proposed by W. Hogg, M.A.)—A uniform rod rests within a rough circle, the plane of which is vertical; find the position of the rod when the friction can only just maintain the equilibrium.

Solution by the PROPOSER.

Let θ = the inclination of the rod to the horizon, when bordering upon motion; 2α = the angle subtended by it at the centre; $\tan\epsilon$ = the coefficient of friction.



Here the resistances μR and μR' must act contrary to the motions that would ensue if the sphere were smooth.

Resolving the forces horizontally, we get

$$R\sin(\alpha-\theta)+\mu R\cos(\alpha-\theta)-R'\sin(\alpha+\theta)+\mu R'\cos(\alpha+\theta)=0,$$

therefore
$$\frac{R'}{R}=\frac{\sin(\alpha-\theta)+\mu\cos(\alpha-\theta)}{\sin(\alpha+\theta)-\mu\cos(\alpha+\theta)} \quad\ldots\ldots\ldots\ldots (1).$$

The moments round G (tending to increase θ) give

$$-R\cos\alpha+\mu R\sin\alpha+R'\cos\alpha+\mu R'\sin\alpha=0,$$

therefore
$$\frac{R'}{R}=\frac{\cos\alpha-\mu\sin\alpha}{\cos\alpha+\mu\sin\alpha} \quad\ldots\ldots\ldots\ldots (2).$$

From equation (1) we have

$$\frac{R-R'}{R+R'}=\frac{\cos\alpha\sin\theta-\mu\cos\alpha\cos\theta}{\sin\alpha\cos\theta+\mu\sin\alpha\sin\theta}=\frac{\cos\alpha}{\sin\alpha}\cdot\frac{\sin\theta-\mu\cos\theta}{\cos\theta+\mu\sin\theta}$$
$$=\frac{\cos\alpha}{\sin\alpha}\cdot\frac{\tan\theta-\mu}{1+\mu\tan\theta} \quad\ldots\ldots\ldots\ldots (3).$$

And from equation (2) we get similarly $\dfrac{R-R'}{R+R'}=\dfrac{\mu\sin\alpha}{\cos\alpha}$ .... (4).

Equating (3) and (4), we obtain $\dfrac{\tan\theta-\mu}{1+\mu\tan\theta}=\mu\tan^2\theta$ ....(5);

$$\therefore\ \tan\theta=\frac{\mu\sec^2\alpha}{1-\mu\tan^2\alpha}=\frac{\mu}{\cos^2\alpha-\mu^2\sin^2\alpha}=\frac{\mu}{(\cos\alpha-\mu\sin\alpha)(\cos\alpha+\mu\sin\alpha)}$$
$$=\frac{\tan\epsilon}{(\cos\alpha-\tan\epsilon\sin\alpha)(\cos\alpha+\tan\epsilon\sin\alpha)}$$
$$=\frac{\cos\epsilon\sin\epsilon}{(\cos\alpha\cos\epsilon-\sin\alpha\sin\epsilon)(\cos\alpha\cos\epsilon+\sin\alpha\sin\epsilon)}$$
$$=\frac{\cos\epsilon\sin\epsilon}{\cos(\alpha+\epsilon)\cos(\alpha-\epsilon)}=\frac{\cos\epsilon\sin\epsilon}{\cos^2\epsilon-\sin^2\alpha}=\frac{\cos\epsilon\sin\epsilon}{\cos^2\epsilon-1+\cos^2\alpha}$$
$$=\frac{2\sin\epsilon\cos\epsilon}{2\cos^2\epsilon-1+2\cos^2\alpha-1}=\frac{\sin 2\epsilon}{\cos 2\epsilon+\cos 2\alpha}.$$

NOTE 1.—Putting $\tan\epsilon$ for μ in equation (5), and it becomes $\tan(\theta-\epsilon)=\tan\epsilon\tan^2\alpha$, which is a better formula for calculation.

NOTE 2.—Resolving the forces vertically, we obtain

$$R\cos(\alpha-\theta)-\mu R\sin(\alpha-\theta)+R'\cos(\alpha+\theta)+\mu R'\sin(\alpha+\theta)-W=0,$$

from which and the preceding relations we get

$$R=\frac{W\cos\theta}{2\cos(\alpha+\epsilon)},\quad\text{and}\quad R'=\frac{W\cos\theta}{2\cos(\alpha-\epsilon)},$$

which are neat expressions.

3367. (Proposed by C. TAYLOR, M.A.)—(1) Interpret the tangential equation $pqr = 0$. (2) Represent by a tangential equation a pair of straight lines regarded as a limit of a conic.

Solution by F. D. THOMSON, M.A.

1. In a paper on "The Limiting Cases of Certain Conics," published in the *Messenger of Mathematics*, Vol. IV., p. 86, I have called attention to the conventions under which we interpret such equations as $xy=0$, $\lambda\mu=0$ in point or line coordinates.

Strictly speaking, every point or Cartesian equation gives an infinite assemblage of points distributed according to some one law; but for convenience we may regard this assemblage as a whole, and call it a locus or curve.

In the same manner, every equation in line or tangential coordinates gives us, strictly speaking, an infinite assemblage of straight lines; but for convenience we may regard this assemblage as a whole, and call it an envelope.

Thus a point or Cartesian equation is sufficiently interpreted if we know how to find the position of every point whose coordinates satisfy it; and a line or tangential equation is sufficiently interpreted if we know how to find the position of every straight line whose coordinates satisfy it.

Thus, if the coordinates of a line be taken to be the perpendiculars λ, μ, ν drawn upon it from three fixed points of reference, the equation $\lambda=0$ gives us, strictly speaking, an infinite number of straight lines all passing through one of the points of reference. But for convenience we may disregard the lines themselves, and think only of their envelope, and thus say that the equation $\lambda = 0$ represents one of the fixed points of reference.

Again, the point equation $xy = 0$ is, for all position-of-point purposes, equivalent to the two lines given by $x=0$, $y=0$; and may, in this sense, be said to be the equation to this system of lines.

In precisely the same manner, the line equation $\lambda\mu = 0$ is, for all position-of-line purposes, equivalent to the two envelopes $\lambda = 0$, $\mu = 0$; and may, in this sense, be said to be the equation to two of the points of reference, all that we mean by this statement being that the coordinates of any line through either of these points satisfy the equation $\lambda\mu = 0$, and each of these lines therefore helps to make up the envelope which the equation represents.

Similarly, the equation $\lambda\mu\nu = 0$ is, as far as regards the position of lines helping to make up the envelope, equivalent to the three envelopes $\lambda=0$, $\mu=0$, $\nu=0$, and in this sense may be interpreted as the equation to the three points of reference; all that we mean by this statement being that the coordinates of any line through either point of reference satisfy the proposed equation, and every such line, therefore, helps to make up the envelope which the equation represents.

2. But although from one point of view the above gives a sufficient interpretation of such equations as $xy = 0$, $\lambda\mu = 0$, $\lambda\mu\nu = 0$, yet it is interesting to regard these equations as derived from others by the vanishing of certain terms, and thus to see how it is that a curve of the second order degenerates into two straight lines, while an envelope of the second class degenerates into two points.

The former case is familiar to all, and need not here be considered.

Take, then, the equation $\lambda\mu = 0$, and let us consider it as being derived

from the equation $\lambda\mu = \kappa PQ$, where P, Q are any two arbitrary points, by κ becoming indefinitely small.

The constant κ will be determined by assigning some fifth tangent RS to the curve (Fig. 1), and it will be seen from the equation that the nearer RS passes to one of the points of reference A, B, the smaller κ becomes. Let RS then move parallel to itself indefinitely close to the point A; then it will be seen, from the figure, that the envelope, if an ellipse, becomes flatter and flatter until it is indefinitely close to the straight line AB; and that if the envelope be an hyperbola, each branch becomes flatter and flatter until the two together make up ultimately what may be called the *complement* of the line AB.

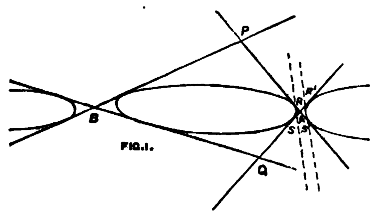


FIG. 1.

It will be seen then, from the figure, how it is that, when κ becomes indefinitely small, every tangent to the envelope passes indefinitely near either to A or B, so that, as explained in (1), we may say that the envelope represented by $\lambda\mu = 0$ is the two points A and B.

3. In the same manner, we may regard the equation $\lambda\mu\nu = 0$ as the limit of the equation $\lambda\mu\nu = \kappa PQR$, when κ is indefinitely small.

Now the latter equation represents an envelope to which the nine lines joining the three points of reference to the arbitrary points P, Q, R, are tangents, and the constant κ would be determined, and the envelope might be traced, by assigning an arbitrary tenth line as a tangent.

But, for convenience of tracing the limiting form of the envelope, it will be simpler to regard $\lambda\mu\nu = 0$ as the limit of $\lambda\mu\nu = \kappa D^3$, where D is an arbitrary point.

Now this equation represents an envelope to which the three points of reference are cusps, and such that the tangents at the cusps meet in the point D. And on assigning an arbitrary tangent A′B′C′, the form of the envelope may be easily obtained. If the position of A′B′C′ be as in Fig. 2, the envelope is a tricusp surrounded by an oval. Suppose A′B′C′ to move parallel to itself until it passes indefinitely close to C, then it will be found that the oval approaches nearer and nearer to the three sides of the triangle, and ultimately coincides with them.

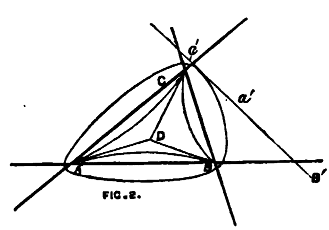


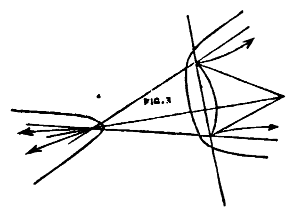
FIG. 2.

It is not so easy to trace what becomes of the tricusp; but it appears that the curvature near the cusps changes more and more rapidly, and each branch becomes flatter and flatter, until the tricusp also moves up indefinitely close to the three sides of the triangle.

The form, however, of the limiting curve may differ from that given in Fig. 2, if the arbitrary tangent A′B′C′ be drawn in a different direction, and then move parallel to itself, as before. Thus the oval may become two infinite branches, two cusps being within one, and the remaining cusp within the other branch; but all these varieties may be obtained from Fig. 2, by drawing an arbitrary straight line to cut the oval and the tricusp, and then projecting this line to infinity. Hence, in any case, the

limiting form of an envelope of the third class, before degenerating into three points, is a curve, all whose points lie indefinitely close either to the perimeter of the triangle formed by these points, or to one of these sides and the complements of the other two, as in Fig. 3.

4. No single tangential equation can be formed to represent two straight lines, regarded as the limit of a conic. For any such equation would be simply the analytical expression of the fact that every tangent passed ultimately indefinitely near to the point of section of the two straight lines, and would therefore be also true for any other pair of straight lines through the same point.



3352. (Proposed by the Rev. A. F. Torry, M.A.)—A chord of a conic subtends a right angle at a fixed point. Show that it envelopes a conic which has the fixed point for focus, and its pole describes another conic, the fixed point having the same polar with respect to all three conics. If the first conic be a circle, the third will be so also.

Solution by F. D. Thomson, M.A.

These theorems may be reduced to simpler ones by reciprocation, but they may be proved analytically as follows:—

1. Take the fixed point as origin, and the coordinate axes parallel to the axes of the given conic. Then the equation to the conic is of the form

$$S \equiv Ax^2 + By^2 + 2Fy + 2Gx + C = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The lines joining the origin to the points where S is met by the chord $\lambda x + \mu y + 1 = 0$ are given by

$$Ax^2 + By^2 - 2(Fy + Gx)(\lambda x + \mu y) + C(\lambda x + \mu y)^2 = 0,$$

or $$(A - 2G\lambda + C\lambda^2)\,x^2 + (B - 2F\mu + C\mu^2)\,y^2 + \&c. = 0.$$

These two lines are perpendicular if

$$A - 2G\lambda + C\lambda^2 + B - 2F\mu + C\mu^2 = 0,$$

or $$\Sigma \equiv C(\lambda^2 + \mu^2) - 2(G\lambda + F\mu) + A + B = 0 \quad\ldots\ldots\ldots\ldots (2).$$

This, therefore, is the *tangential* equation to the envelope of the chord, and represents a conic of which the origin is one focus, while the *other* focus is at the point whose *tangential* equation is

$$2G\lambda + 2F\mu - (A + B) = 0,$$

that is, the point given in Cartesian coordinates by

$$\frac{x}{2G} = \frac{y}{2F} = \frac{-1}{A+B}.$$

The Cartesian equation corresponding to (2), that is, to

$$(C,\ C,\ A+B,\ -F,\ -G,\ 0 \between \lambda, \mu, 1)^2 = 0,$$

is $\quad (C(A+B)-F^2,\ C(A+B)-G^2,\ C^2,\ FC,\ GC,\ FG\between x, y, 1)^2 = 0;$
and therefore the polar of the origin is $\quad Fy+Gx+C=0.$

2. Again, let (x', y') be the pole of the line $\lambda x+\mu y+1=0$ with respect to S; then $\lambda x+\mu y+1=0$ is identical with

$$x(Ax'+G)+y(By'+F)+Fy'+Gx'+C=0,$$

therefore
$$\frac{\lambda}{Ax'+G}=\frac{\mu}{By'+F}=\frac{1}{Fy'+Gx'+C}.$$

Dropping accents and substituting for λ, μ in (2), we have

$$C[(Ax+G)^2+(By+F)^2]-2[G(Ax+G)+F(By+F)][Fy+Gx+C]$$
$$+(A+B)[Fy+Gx+C]^2=0 \quad \text{.......................} (3),$$

or $\quad [A^2C+(B-A)G^2]x^2+[B^2C+(A-B)F]y^2$
$$+[(A+B)C-F^2-G^2][2Gx+2Fy+C]=0,$$

the equation to the locus, which becomes a circle if $A=B$ or if the original conic is a circle. The polar of the origin with respect to (3) is

$$Fy+Gx+C=0.$$

3200. (Proposed by the Rev. G. H. Hopkins, M.A.)—If a positive integral value of z, in the equation $x^2=y^2+z^2$, be $2a_1a_2a_3\ldots a_n$, ($a_1, a_2, a_3, \ldots a_n$ being prime numbers), then the number of integral values of x or y which correspond to this value of z will be $\frac{1}{2}(3^n-1)$.

Solution by the Proposer.

The general forms of x, y, z in the solution of $x^2=y^2+z^2$ are $p(\alpha^2+\beta^2)$, $p(\alpha^2-\beta^2)$, $p.2\alpha\beta$; α, β, and p being positive integers.

First, assume p to be unity, then α and β will have such values that their product may be $a_1a_2a_3\ldots a_n$; and for different values of α and β we shall have different values of x and y.

Thus the value of z may be thrown into the form $2(a_1a_2a_3\ldots a_n)\times(1)$; the corresponding values of x and y will be $(a_1a_2a_3\ldots a_n)^2+1$ and $(a_1a_2a_3\ldots a_n)^2-1$.

Or α may equal any $n-1$ of the factors, β the remaining factor. Evidently there will be n solutions arising from this.

Again, α may equal any $n-2$ of the factors, and β the remaining two factors. There will be $\frac{n(n-1)}{1.2}$ solutions arising from this arrangement.

Continuing in this way, the total number of solutions (when $p=1$) would appear to be $\left(1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\&c.+1\right)$ or 2^n;

but this will give double the true number, for it includes such solutions as
$$\alpha\beta = (a_1 a_2 \ldots a_h)(a_{h+1} a_{h+2} \ldots a_n) \quad \text{and} \quad \alpha\beta = (a_{h+1} a_{h+2} \ldots a_n)(a_1 a_2 \ldots a_h),$$
which are obviously the same; therefore the true number is $\frac{1}{2} \times 2^n$.

Secondly, when p is not unity, p can equal any of the factors or any combination of factors of z (excepting the factor 2).

If $p = a_n$, then the number of solutions arising from the remaining $n-1$ factors will be $\frac{1}{2} \times 2^{n-1}$.

But p can equal any of the factors taken separately; therefore the number of solutions when p equals any single one of the factors will be $\frac{1}{2}n \times 2^{n-1}$.

Again, p may equal any two of the factors; the number of solutions arising therefrom will be $\frac{1}{2} \cdot \frac{n(n-1)}{1.2} \times 2^{n-2}$; and so on.

If p equal any $n-1$ of the factors, there will be n solutions, or $\frac{1}{2}n \times 2$.

There will be no solution when p is equal to all of the factors together, for then y would equal zero; therefore the whole number will be
$$\frac{1}{2}\left\{2^n + n \cdot 2^{n-1} + \frac{n(n-1)}{1.2} 2^{n-2} + \&c. + n \cdot 2\right\} \quad \text{or} \quad \frac{1}{2}\left\{(2+1)^n - 1\right\},$$
that is, $\frac{1}{2}(3^n - 1)$.

The number 19882, being the double of a prime number, has but one solution, which is 98823482, 98823480.

The four solutions of 19886 $(2 \times 61 \times 163)$ will be
$$\text{for } x, \quad 61^2 + 163^2, \quad (61 \times 163)^2 + 1, \quad 61(163^2 + 1), \quad 163(61^2 + 1);$$
$$\text{for } y, \quad 163^2 - 61^2, \quad (61 \times 163)^2 - 1, \quad 61(163^2 - 1), \quad 163(61^2 - 1).$$

The 121 values of x or y for 30030 can be found without much difficulty.

3384. (Proposed by R. W. GENESE, B.A.)—Show that the product of the three normals that can be drawn from a point P on a central conic is equal to twice PG . PM . PM′, where PM, PM′ are perpendiculars on the directrices, and PG is the part of the normal at P intercepted by the major axis.

Solution by J. J. WALKER, M.A.

Referring to my solution of Question 3096 (*Reprint*, Vol. XIV., p. 38), if (x, y) be on the ellipse, the equation (4) will be divisible by $x' - x$. The result is
$$c^4 x'^3 - (a^4 - b^4) x x'^2 + a^4 (2b^2 - a^2) x' + a^6 x = 0,$$
where x' is the abscissa of one of three points, normals at which meet in (x, y). If r be the length of one of these normals,
$$r^2 = \frac{a^2 (x' - x)^2 (a^2 - e^2 x'^2)}{b^2 x'^2};$$
and if x_1', x_2', x_3' be the roots of the cubic above, by pursuing steps precisely

analogous to those adopted in the solution referred to, it will be found that

$$(a^2-e^2x_1'^2)(a^2-e^2x_2'^2)(a^2-e^2x_3'^2) = \frac{a^{10}b^4(a^2-e^2x^2)}{c^4},$$

also that $$(x_1'-x)^2(x_2'-x)^2(x_3'-x)^2 = \frac{4a^4b^4x^2(a^2-e^2x^2)}{c^8};$$

and since $x_1'^2x_2'^2x_3'^2 = \dfrac{a'^2x^2}{c^8}$, it follows that

$$r_1^2r_2^2r_3^2 = \frac{4a^2b^2}{c^4}(a^2-e^2x^2)^3 = 4\text{PG}^2.\text{PM}^2.\text{PM}'^2,$$

since $$\text{PG}^2 = \frac{b^2}{a^2}(a^2-e^2x^2), \quad \text{and} \quad \text{PM}.\text{PM}' = \frac{a^2-e^2x^2}{e^2}.$$

3393. (Proposed by R. Tucker, M.A.)—K, K′ are any two points upon an ellipse; the tangents at these points meet in P, and the normals meet the major axis in N, N′; prove (1) that ∠ KPN = ∠ K′PN′; and (2) that if O, O′ be the centres of curvature at the same points, and ρ, ρ' the radii, then

$$\tan^3 \text{KPO} : \tan^3 \text{K'PO'} = \rho^2 : \rho'^2.$$

I. *Solution by the* Proposer.

1. If ϕ, ϕ' be the eccentric angles for K, K′, then the equations to PK, PK′ are $$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1, \quad \frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' = 1 \quad \text{.........}(1, 2);$$

therefore $$\frac{x}{a} = \frac{\cos\frac{1}{2}(\phi'+\phi)}{\cos\frac{1}{2}(\phi'\sim\phi)}, \quad \text{and} \quad \frac{y}{b} = \frac{\sin\frac{1}{2}(\phi'+\phi)}{\sin\frac{1}{2}(\phi'\sim\phi)};$$

hence $$(\text{PK})^2 = \tan^2\tfrac{1}{2}(\phi'\sim\phi)(a^2\sin^2\phi + b^2\cos^2\phi),$$

and $$\text{PK} = b'\tan\tfrac{1}{2}(\phi'\sim\phi),$$

where b' is the semi-conjugate diameter to CK. (Salmon's *Conics*, Art. 173.)

Similarly, $$\text{PK}' = b''\tan\tfrac{1}{2}(\phi'\sim\phi).$$

Again (Salmon's *Conics*, Art. 181), $\text{KN} = \dfrac{bb'}{a}$, $\text{K'N'} = \dfrac{bb''}{a}$;

therefore $$\text{PK} : \text{KN} = \text{PK}' : \text{K'N'},$$
that is, the angle KPN = K′PN′.

2. Again, $$\tan \text{KPO} = \frac{\rho}{\text{PK}},$$

therefore $$\frac{\tan \text{KPO}}{\tan \text{K'PO'}} = \frac{\rho}{\rho'}\cdot\frac{\text{PK}'}{\text{PK}} = \left(\frac{b'}{b''}\right)^3\cdot\frac{b''}{b'},$$ (Salmon's *Conics*, Art. 292);

therefore $$\frac{\tan^3 \text{KPO}}{\tan^3 \text{K'PO'}} = \left(\frac{\rho}{\rho'}\right)^2.$$

[The November number of the *Nouvelles Annales de Mathématiques* was not published until the end of March, 1871, a long time after Quest. 3393 had been in the Editor's hands. Hence it appears that M. G. Dostor's question on p. 527 of the *Annales*, and Quest. 3393, which is identical with it, were obtained independently of each other.]

3173. (Proposed by J. J. Walker, M.A.)—In any spherical triangle ABC, let D be the middle point of the arc BC, and E another point in BC such that the angle BAE is equal to the angle DAC; also let AF be the arc through A perpendicular to BC; prove that

$$\frac{\cos \text{AEB}}{\cos \text{ADB}} = -\cos(\text{B}+\text{C}), \quad \tan \tfrac{1}{2}\text{DAE} = \frac{\tan \frac{1}{2}(b-c)}{\tan \frac{1}{2}(b+c)} \tan \tfrac{1}{2}\text{A} \ldots\ldots (1, 2),$$

$$\tan \text{EAF} = \frac{\sin^2 \frac{1}{2}c - \sin^2 \frac{1}{2}b}{\sin^2 \frac{1}{2}b \cos^2 \frac{1}{2}c + \sin^2 \frac{1}{2}c \cos^2 \frac{1}{2}b} \cot(\text{B}+\text{C}) \ldots\ldots\ldots (3),$$

$$\cot \text{ADB} = \frac{\cot \text{C} - \cot \text{B}}{2 \cos \frac{1}{2}a}, \quad \frac{\tan \text{AE}}{\tan \text{DE}} = \frac{\sin b \sin c}{1 - \cos b \cos c} \ldots\ldots (4, 5).$$

Solution by the Proposer.

(1) and (2). $\dfrac{\sin \text{CAF}}{\sin \text{BAF}} = \dfrac{\cos \text{C}}{\cos \text{B}}$, therefore if the arc AG bisects the angle BAC (or DAE), we have

$$\tan \text{GAF} = \tan \tfrac{1}{2}(\text{B}+\text{C}) \tan \tfrac{1}{2}(\text{B}-\text{C}) \tan \tfrac{1}{2}\text{A}.$$

Again, $$\frac{\sin \text{DAB}}{\sin \text{DAC}} = \frac{\sin \text{B}}{\sin \text{C}};$$

therefore

$$\tan \text{DAG} = \frac{\tan \frac{1}{2}(\text{B}-\text{C})}{\tan \frac{1}{2}(\text{B}+\text{C})} \tan \tfrac{1}{2}\text{A}, \quad \text{and} \quad \frac{\tan \text{GAF}}{\tan \text{DAG}} = \tan^2 \tfrac{1}{2}(\text{B}+\text{C});$$

whence we obtain

$$\frac{\sin(\text{GAF}-\text{DAG})}{\sin(\text{GAF}+\text{DAG})} = \frac{\tan^2 \frac{1}{2}(\text{B}+\text{C}) - 1}{\tan^2 \frac{1}{2}(\text{B}+\text{C}) + 1}, \quad \text{or} \quad \frac{\sin \text{EAF}}{\sin \text{DAF}} = -\cos(\text{B}+\text{C}).$$

But from the right-angled triangles EAF, DAF, $\dfrac{\sin \text{EAF}}{\sin \text{DAF}} = \dfrac{\cos \text{AEB}}{\cos \text{ADB}}$.

(3). Tan EAF = tan (GAF − GAE or DAG) = (from above)

$$\frac{\tan \frac{1}{2}(\text{B}-\text{C}) \left\{\tan^2 \frac{1}{2}(\text{B}+\text{C}) - 1\right\} \tan \frac{1}{2}\text{A}}{\tan \frac{1}{2}(\text{B}+\text{C}) \left\{\tan^2 \frac{1}{2}(\text{B}-\text{C}) \tan^2 \frac{1}{2}\text{A} + 1\right\}},$$

which is easily reduced to $\dfrac{\sin \text{A} \sin(\text{C}-\text{B})}{1 + \cos \text{A} \cos(\text{B}-\text{C})} \cot(\text{B}+\text{C})$,

and, by means of the known formula $\dfrac{\cos \frac{1}{2}(\text{A}+\text{B}-\text{C})}{\cos \frac{1}{2}(\text{A}-\text{B}+\text{C})} = \dfrac{\tan \frac{1}{2}c}{\tan \frac{1}{2}b}$, this may be transformed into $\dfrac{\cos b - \cos c}{1 - \cos b \cos c} \cot(\text{B}+\text{C})$.

But $$\frac{\cos b - \cos c}{1 - \cos b \cos c} = \frac{\sin^2 \frac{1}{2}c - \sin^2 \frac{1}{2}b}{\sin^2 \frac{1}{2}b \cos^2 \frac{1}{2}c + \sin^2 \frac{1}{2}c \cos^2 \frac{1}{2}b}.$$

Formulæ (1), (2), (3) are the analogues of the three referring to the plane triangle given in Quest. 3060 (*Reprint*, Vol. XIII., p. 67).

(4). Let p = AF, then $\sin \text{BF} = \tan p \cot \text{C}$, $\sin \text{CF} = \tan p \cot \text{B}$; therefore $2 \cos \frac{1}{2}a \sin \text{DF} = \tan p (\cot \text{C} - \cot \text{B})$; but $\sin \text{DF} = \tan p \cot \text{D}$; therefore, &c.

(5). $\dfrac{\tan \text{GAF}}{\cot \text{DAG}} = \tan^2 \frac{1}{2}(\text{B}-\text{C}) \tan^2 \frac{1}{2}\text{A}$, from above;

therefore $\dfrac{\cot \mathrm{DAG} - \tan \mathrm{GAF}}{\cot \mathrm{DAG} + \tan \mathrm{GAF}}$ or $\dfrac{\cos \mathrm{DAF}}{\cos \mathrm{EAF}} = \dfrac{1 - \tan^2 \frac{1}{2}(\mathrm{B}-\mathrm{C}) \tan^2 \frac{1}{2}\mathrm{A}}{1 + \tan^2 \frac{1}{2}(\mathrm{B}-\mathrm{C}) \tan^2 \frac{1}{2}\mathrm{A}}$

$$= \frac{\cos(\mathrm{B}-\mathrm{C}) + \cos \mathrm{A}}{1 + \cos \mathrm{A} \cos(\mathrm{B}-\mathrm{C})} = 2\,\frac{\cos \frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C}) \cos \frac{1}{2}(\mathrm{A}-\mathrm{B}+\mathrm{C})}{\cos^2 \frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C}) + \cos^2 \frac{1}{2}(\mathrm{A}-\mathrm{B}+\mathrm{C})}.$$

But if R be the radius of the circle circumscribing ABC,

$$\cot \tfrac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C}) = \tan \tfrac{1}{2}c \cot \mathrm{R}, \quad \cos \tfrac{1}{2}(\mathrm{A}-\mathrm{B}+\mathrm{C}) = \tan \tfrac{1}{2}b \cot \mathrm{R},$$

whence $$\frac{\cos \mathrm{DAF}}{\cos \mathrm{EAF}} = \frac{2 \tan \frac{1}{2}b \tan \frac{1}{2}c}{\tan^2 \frac{1}{2}b + \tan^2 \frac{1}{2}c} = \frac{\sin b \sin c}{1 - \cos b \cos c};$$

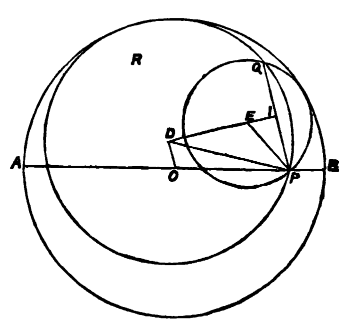
while from the right-angled triangles DAF, EAF, $\dfrac{\cos \mathrm{DAF}}{\cos \mathrm{EAF}} = \dfrac{\tan \mathrm{AE}}{\tan \mathrm{AD}}$.

[A solution of parts (1), (2), (3) of the question has been given on p. 97, of Vol. XV. of the *Reprint*.]

1843. (Proposed by the Editor.)—Three points being taken at random within a circle, find the chance that the circle drawn through them will lie wholly within the given circle.

3164. (Proposed by W. S. B. Woolhouse, F.R.A.S.)—Innumerable triads of points are taken at random within a given circle, and each separate triad is distinguished by the colour black or red, according as the circle drawn through the points may respectively happen to lie wholly within, or to pass partly beyond, the circumference of the given circle; show that, in the aggregate collection of points thus made, the greatest density of the black, and the least density of the red, will be exhibited at a distance of two-thirds of the radius from the centre of the given circle.

Solution by W. S. B. Woolhouse, F.R.A.S.



Of the original Question 1843 several solutions, more or less imperfect, have been given in the *Educational Times* since the first proposal of this remarkable question by the Editor. (See *Reprint*, Vol. VIII., pp. 90—92, and Vol. XIII., pp. 17—21, and pp. 95—99.) Amongst them is an approximate solution by myself, besides a Note explanatory of the principle that should be adopted in a strict discussion of the proposed condition. It was reserved for the indefatigable talent and perseverance of Mr. Watson to accomplish an accurate and effective investigation of

the problem. It would be impossible to compliment Mr. Watson too highly on the ability he has so successfully displayed in the valuable and interesting solution published in Vol. XIII. (pp. 95—99) of the *Reprint*.

I have now to annex the following solution to the Question 3164, which was proposed expressly for the purpose of giving a supplementary discussion that completes the development of the subject by finally eliciting the law of distribution of the points. I shall, for the sake of uniformity and clearness, follow the same notation as Mr. Watson. His solution of Question 1843 would have included, in substance at least, the implied relations which subsist with reference to the distribution of the points, excepting that he has somewhat diverted the natural course of the successive integrations, in order to obtain a greater facility in getting out his final result. For brevity, in establishing the theorem in Quest. 3164, I shall make use of some of Mr. Watson's principal formulæ, the whole of which are worked out with his usual skill and accuracy.

According to the prescribed notation, we have

$$OB=1,\ OP=x,\ PQ=y,\ \angle OPQ=\phi,\ \angle OPD=a,\ \angle OPE=\beta,$$
$$1-x^2=m,\ 1-x\sin\phi=Y,\ \text{and}\ 1+x\sin\phi=Y_1.$$

The probability in Quest. 1843 depends upon the successive integrations

$$p=\frac{2}{\pi^2}\iiint x\,dx\,d\phi\,dy\,y^3\Big\{(\phi-a)\sec^2(\phi-a)-(\phi-\beta)\sec^2(\phi-\beta)$$
$$+\tan(\phi-a)-\tan(\phi-\beta)\Big\},$$

which are now to be taken throughout in their natural order, that is to say, successively with respect to the variables (y, a), ϕ, x; where y, a are mutually considered as implicit functions. The integration with respect to x should be thus deferred and made the last operation. In some of the terms Mr. Watson has integrated finally with respect to ϕ, the integration with respect to x having been performed previously. What we have now to effect is to suitably modify the order of this procedure.

Mr. Watson's final expression of the probability consists of four terms,

viz. $$p=\frac{1}{\pi^2}\Big\{(8)+(12)+(15)+(18)\Big\},$$

where $$\frac{d(8)}{x\,dx}=\tfrac{5}{4}\pi^2\Big\{(1-x^2)^{\frac32}-(1-x^2)\Big\}=\tfrac{5}{4}\pi^2\,(m^{\frac32}-m),$$

and $$\frac{d(12)}{x\,dx}=\int_0^{\frac12\pi}d\phi\,\Big\{(10)+(11)\Big\}.$$

Also, as (15) originates exclusively from the last line of Mr. Watson's formula (5), we have,

$$\frac{d(15)}{x\,dx}=\tfrac{5}{2}m^{\frac12}x^2\int d\phi\,\sin 2\phi\int da\,\tan^{-1}\Big\{\Big(\frac{1+x}{1-x}\Big)^{\frac12}\tan\tfrac12 a\Big\}$$
$$=\tfrac{5}{4}mx^2\int d\phi\,\sin 2\phi\int da\int_0^{a}\frac{da}{1-x\cos a};$$

and $$\frac{d(18)}{x\,dx}=-\pi\frac{m^{\frac12}}{6x^2}\int_0^{\frac12\pi}d\phi\,\Big\{2m\,(1-4x^2)\cos 2\phi-x^2\,(6+9x^2)\Big\}.$$

To proceed with the integrations indicated in these last three formulæ,

we have first $(10)+(11) = -\frac{m^2\pi}{6}\left(\frac{m}{Y^2}+\frac{m}{Y_1^2}\right)+\left(\frac{5}{Y}+\frac{5}{Y_1}\right)$

$$+\frac{m\pi}{3x^2}\left\{1+3x^2-4x^4-(2-9x^2-8x^4)\sin^2\phi\right\}.$$

Now we have

$$\int\frac{d\phi}{Y_1}=\int\frac{d\phi}{1+x\sin\phi}=\frac{2}{m^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\frac{\phi+\alpha}{2}\right\}=\frac{2}{m^{\frac{1}{2}}}\tan^{-1}t_1;$$

$$\int\frac{d\phi}{Y}=\int\frac{d\phi}{1-x\sin\phi}=\frac{2}{m^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}\tan\frac{\phi-90^\circ}{2}\right\}=\frac{2}{m^{\frac{1}{2}}}\tan^{-1}t_2;$$

$$\int\frac{d\phi}{Y_1^2}=\int\frac{d\phi}{(1+x\sin\phi)^2}=\frac{1}{m}\left(\int\frac{d\phi}{1+x\sin\phi}+\frac{x\cos\phi}{1+x\sin\phi}\right);$$

$$\int\frac{d\phi}{Y^2}=\int\frac{d\phi}{(1-x\sin\phi)^2}=\frac{1}{m}\left(\int\frac{d\phi}{1-x\sin\phi}-\frac{x\cos\phi}{1-x\sin\phi}\right).$$

Also $\tan^{-1}t_1+\tan^{-1}t_2=\tan^{-1}(m^{\frac{1}{2}}\tan\phi)$, and $\int_0^{\frac{1}{2}\pi}d\phi\sin^2\phi=\frac{1}{4}\pi$;

therefore we obtain

$$\int d\phi\left\{(10)+(11)\right\}=-\frac{m^{\frac{3}{2}}\pi}{3}\left\{6\tan^{-1}(m^{\frac{1}{2}}\tan\phi)-\frac{x^2\cos\phi\sin\phi}{1-x^2\sin^2\phi}\right\}+\frac{5m\pi^2}{4};$$

and $\frac{d\,(12)}{x\,dx}\int_0^{\frac{1}{2}\pi}d\phi\left\{(10)+(11)\right\}=-m^{\frac{3}{2}}\pi^2+\frac{5m\pi^2}{4}=\pi^2(\frac{5}{4}m-m^{\frac{3}{2}})$.

Again, we have

$$\int_0^\alpha\frac{d\alpha}{1-x\cos\alpha}=\int_0^\alpha d\alpha\,(1+x\cos\alpha+x^2\cos^2\alpha+x^3\cos^3\alpha\,\ldots\ldots)$$

$$=\alpha+x\sin\alpha+x^2(\tfrac{1}{2}\cos\alpha\sin\alpha+\tfrac{1}{2}\alpha)+x^3(\tfrac{1}{3}\cos^2\alpha+\tfrac{2}{3})\sin\alpha$$
$$+x^4\left\{(\tfrac{1}{4}\cos^3\alpha+\tfrac{3}{8}\cos\alpha)\sin\alpha+\tfrac{3}{8}\alpha\right\}\ldots\ldots;$$

therefore $\int d\alpha\int_0\frac{d\alpha}{1-x\cos\alpha}=\frac{1}{2}\alpha^2(1+\frac{1}{2}x^2+\frac{3}{8}x^4\ldots)$

$$-x\cos\alpha-\tfrac{1}{4}x^2\cos^2\alpha-x^3(\tfrac{1}{9}\cos^3\alpha+\tfrac{2}{3}\cos\alpha)\ldots\ldots,$$

which, between the limits $(\phi+\frac{1}{2}\pi,\ \phi-\frac{1}{2}\pi)$, gives

$$\frac{\pi\phi}{(1-x^2)^{\frac{1}{2}}}+2x\sin\phi+2x^3(\tfrac{1}{9}\sin^3\phi+\tfrac{2}{3}\sin\phi)\ldots\ldots;$$

hence we have $\int d\phi\sin 2\phi\int d\alpha\int_0^\alpha\frac{d\alpha}{1-x\cos\alpha}$

$$=2\int d\sin\phi\,.\,\sin\phi\left\{\frac{\pi\phi}{(1-x^2)^{\frac{1}{2}}}+2x\sin\phi+2x^3(\tfrac{1}{9}\sin^3\phi+\tfrac{2}{3}\sin\phi)\ldots\ldots\right\}$$

$$=\frac{\pi}{m^{\frac{1}{2}}}(\tfrac{1}{2}\cos\phi\sin\phi-\tfrac{1}{2}\phi+\phi\sin^2\phi)+4x\frac{\sin^3\phi}{3}+4x^3(\tfrac{1}{45}\sin^5\phi+\tfrac{2}{9}\sin^3\phi)\ldots;$$

which, taken between the limits 0 and π, gives $-\frac{\pi^2}{2m^{\frac{1}{2}}}$. Whence we find

$$\frac{d\,(15)}{x\,dx}=(\tfrac{5}{4}mx^2)\times\left(-\frac{\pi^2}{2m^{\frac{1}{2}}}\right)=-\tfrac{5}{8}\pi^2x^2m^{\frac{1}{2}}=-\tfrac{5}{8}\pi^2(m^{\frac{1}{2}}-m^{\frac{3}{2}});$$

also $-\int_0^{\frac{1}{2}\pi}d\phi\left\{2m(1-4x^2)\cos 2\phi-x^2(6+9x^2)\right\}=\frac{1}{2}\pi x^2(6+9x^2)$;

therefore $$\frac{d(18)}{x\,dx} = \pi\frac{m^{\frac{1}{2}}}{6x^2}\times\tfrac{1}{2}\pi x^2(6+9x^2) = \pi^2 m^{\frac{1}{2}}\,\frac{2+3x^2}{4}$$
$$= \pi^2 m^{\frac{1}{2}}\,\frac{5-3m}{4} = \tfrac{1}{4}\pi^2\left(5m^{\frac{1}{2}}-3m^{\frac{3}{2}}\right).$$

Hence we have $$\frac{d(8)}{x\,dx} = \tfrac{5}{4}\pi^3\left(m^{\frac{1}{2}}-m\right),\qquad \frac{d(12)}{x\,dx} = \pi^2\left(\tfrac{5}{4}m-m^{\frac{3}{2}}\right),$$
$$\frac{d(15)}{x\,dx} = -\tfrac{5}{8}\pi^2\left(m^{\frac{1}{2}}-m^{\frac{3}{2}}\right),\qquad \frac{d(18)}{x\,dx} = \tfrac{1}{4}\pi^2\left(5m^{\frac{1}{2}}-3m^{\frac{3}{2}}\right).$$

Adding together these four values, and dividing out the π^2, we obtain
$$\frac{dp}{xdx} = \frac{1}{\pi^2}\left\{\frac{d(8)}{x\,dx}+\frac{d(12)}{x\,dx}+\frac{d(15)}{x\,dx}+\frac{d(18)}{x\,dx}\right\} = \tfrac{3}{8}\,m^{\frac{1}{2}}(5-3m);$$

therefore $$p = \tfrac{3}{16}\int_0^1 dm\left(5m^{\frac{1}{2}}-3m^{\frac{3}{2}}\right) = \tfrac{3}{16}\left(\tfrac{10}{3}-\tfrac{6}{5}\right) = \tfrac{2}{5},$$

which agrees with Mr. Watson's result.

From the value of $\frac{dp}{x\,dx}$ we at once deduce the sought law of density of the particular points P′, which appertain to the cases favourable to the proposed condition. Thus, according to the investigation, the uniform density of the *total points* taken on the circle is *unity*, since the area π is made to represent the number of points P taken upon it; and it follows that the number of *favourable points* P′ on the circle is πp, and that the number on the elementary annulus $(2\pi x\,dx)$ is πdp. Hence the density of the favourable or *black points* P′ on this annulus is

$$\delta = \frac{\text{points}}{\text{area}} = \frac{\pi dp}{2\pi x\,dx} = \frac{dp}{2x\,dx} = \tfrac{3}{16}\,m^{\frac{1}{2}}(5-3m) = \tfrac{3}{16}(2+3x^2)(1-x^2)^{\frac{1}{2}},$$

and, of course, the density of the *red points* is $1-\delta$. At the distance from the centre at which the density δ is the greatest possible, we shall have

$$u = 5m^{\frac{1}{2}}-3m^{\frac{3}{2}}\text{ a maximum,}\quad\text{therefore}\quad \frac{du}{dm} = \tfrac{5}{2}m^{-\frac{1}{2}}-\tfrac{9}{2}m^{\frac{1}{2}} = 0;$$

whence we have $\qquad m\,(=1-x^2) = \tfrac{5}{9};$ and $x = \tfrac{2}{3}$,
which establishes the theorem enunciated in Question 3164.

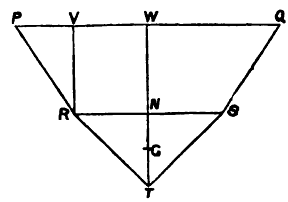
Substituting $\frac{2}{3}$ for x, the calculated maximum density is $\delta = \frac{5}{24}5^{\frac{1}{2}} = \cdot466$. As a practical exhibition of the law of distribution, a few numerical values of the densities are tabulated in the margin. There are no black points at the circumference of the circle, but the density in this case, although the least in value, does not partake of the character of a minimum.

Distance (x)	Black points (δ)	Red points ($1-\delta$)
·0	·3750	·6250
·1	·3787	·6213
·2	·3895	·6105
·3	·4060	·5940
·4	·4262	·5738
·5	·4465	·5535
·6	·4620	·5380
·7	·4646	·5354
·8	·4410	·5590
·9	·2216	·7784
1·0	·0000	1·0000

NOTE.—In all the various problems appertaining to random points, the law of distribution of the particular points which fulfil a given condition, or relation, may be determined in like manner.

3449. (Proposed by T. T. Wilkinson, F.R.A.S.)—The distance of two horizontal points is 20 inches; to these points the ends of two strings, each 12 inches long, are attached; their other ends are fastened to two uniform heavy rods, each 8 inches long, which revolve freely round a hinge at the other extremity. Find the angle which the rods make with each other when at rest.

Solution by the Proposer.



Let P, Q be the two points in the given horizontal line; RT, TS the equal uniform heavy rods, each 8 inches in length, and moving freely round the hinge at T. Draw TW, RV perpendicular to PQ; let RS cut TW in N; and suppose G to be the common centre of gravity of the system. Then, since the rods are heavy, and the strings of no weight, G must lie in the line which joins the middle points of TR and TS, and TG = GN. But RN = NS = VW; RV = NW; also PW = WQ; and WG will be a maximum when the system is at rest. Now let PR = QS = a = 12; PW = b = 10; RT = TS $-$ c = 8; and put the angle RTN = ϕ. Then we have

$$\text{RN} + \text{VW} = c \sin\phi, \quad \text{TN} = c\cos\phi, \quad \text{RV} = \text{NW} = \left\{a^2-(b-c\sin\phi)^2\right\}^{\frac{1}{2}};$$

$$\text{NW} + \tfrac{1}{2}\text{TN} = \text{GW} = \left\{a^2-(b-c\sin\phi)^2\right\}^{\frac{1}{2}} + \tfrac{1}{2}c\,(1-\sin^2\phi)^{\frac{1}{2}} = \text{a maximum}.$$

Equating the differential of this to zero, and reducing, we have

$$3\sin^4\phi - 6bc\sin^3\phi - (4c^2-3b^2-a^2)\sin^2\phi + 8bc\sin\phi - 4b^2 = 0.$$

Inserting numbers, and applying Horner's method, we find $\sin\phi = \cdot 654375$; therefore $\phi = 40^\circ\ 52'$, and the angle RTS = $81^\circ\ 44'$, as required.

3459. (Proposed by the Rev. Dr. Booth, F.R.S.)—In any plane curve whose equation is $\mathrm{F}(x, y) = 0$, let θ be the angle between the perpendicular from the origin on the tangent and the radius-vector to the point of contact. Show (1) that

$$\tan\theta = \left(\frac{d\mathrm{F}}{dx}y - \frac{d\mathrm{F}}{dy}x\right) \div \left(\frac{d\mathrm{F}}{dx}x + \frac{d\mathrm{F}}{dy}y\right);$$

and (2), that by applying this to $\mathrm{A}x^2 + \mathrm{A}_1 y^2 + 2\mathrm{B}xy = 1$, we have

$$\tan\theta = (\mathrm{A}-\mathrm{A}_1)\,xy + \mathrm{B}\,(y^2-x^2).$$

Solution by Stephen Watson; *the* Proposer; *and others.*

1. Let α and β be the respective angles which the radius vector and the perpendicular on the tangent make with the axis of x; then we have

$$\tan\alpha = \frac{y}{x}, \text{ and } \tan\beta = \frac{d\mathrm{F}}{dy} \div \frac{d\mathrm{F}}{dx};$$

$$\text{hence}\quad \tan\theta = \tan(\alpha-\beta) = \frac{\frac{y}{x} - \left(\frac{dF}{dy} \div \frac{dF}{dx}\right)}{1 + \frac{y}{x}\left(\frac{dF}{dy} \div \frac{dF}{dx}\right)} = \frac{\frac{dF}{dx}y - \frac{dF}{dy}x}{\frac{dF}{dx}x + \frac{dF}{dy}y}.$$

2. In the particular case, the formula gives

$$\tan\theta = \frac{(A-A_1)\,xy + B\,(y^2-x^2)}{Ax^2 + A_1y^2 + 2Bxy} = (A-A_1)\,xy + B\,(y^2-x^2).$$

On Experimental Probability. *By* Elizabeth Blackwood.

The object of this paper is to describe a simple experimental method, by means of which *all* the readers of this journal, mathematical and non-mathematical, may obtain approximate solutions to some of the probability questions which appear in it from time to time, and so test the correctness of the more scientific solutions supplied by the mathematical contributors. As an example of the class of problems to which I refer, I would here propose for solution the following question:—

"A point is taken at random on a window consisting of nine equal square panes, and through this point a line is drawn in a random direction. What are the respective chances of the line so drawn cutting one, two, three, four, or five panes?"

Experimentally this may be solved as follows. Take a circular piece of white card-board; draw a straight line across the centre on one side, and through the centre on the opposite side insert a pin. Move the card-board round this pin as a pivot till the stiffness is so far gone that when the pin is held by the point head downwards, a slight jerk of the finger will make the card-board revolve easily and rapidly. This may be a rude contrivance, but it will answer the purpose. Now draw a figure on paper representing the window; twirl the card-board, and while it is spinning mark with a pencil a point anywhere on the figure. If the rotatory motion of the card-board still continues, arrest it suddenly, so as to economize time; then place the card-board on the paper so that the pin's head shall fall on the pencil mark, or near it, extreme accuracy upon this point not being at all necessary. Observe now how many panes the line drawn across the centre of the card-board would cut through if produced; and register accordingly. Mark another point with the pencil, and proceed as before; and so on as long as you please; taking care, however, to distribute the points as impartially as you can over the surface of the figure. It may be worth while remarking that, as the breadth of the lines separating the panes is assumed to be infinitesimal, it is morally certain that the random line cannot pass through any point at which four panes meet. Whenever, therefore, it seems to do so, we may assume as certain that it cuts *three* of those panes, and *not two*, as we might hastily suppose. When the random line appears to pass through a point at which *two* panes meet, we are then left in real doubt as to the number of panes cut. In this case we may register one-half to each of the two doubtful numbers; or we may leave this trial out of account altogether.

Rough and unscientific as this method of experimental verification may

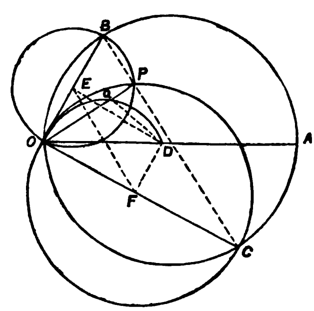
appear, I believe it to be based upon a sound principle; and it certainly has two important advantages, namely, that anyone can apply it, and that it enables us to make a great many trials in a very short time. I have already applied the method to the question which I have given as an example, and the following are my results:—One pane 58 times, two panes 98, three panes 391½, four panes 405, five panes 197½; total number of trials 1150. I therefore conclude that the respective chances are approximately ·050, ·085, ·340, ·352, ·173.

I would suggest that other readers and contributors of the *Educational Times* should perform similar experiments, and send their results to the Mathematical Editor. It would be very interesting to compare the results of, say twenty or thirty thousand trials, with the numerical results obtained from an exact solution of this or similar problems.

In a future paper I hope to give a description of another experimental method, of a totally different kind from this, and capable of a much wider application.

3445. (Proposed by R. Tucker, M.A.)—Two chords of a circle, drawn from a fixed point on the circumference, contain a given angle; prove that circles on the chords as diameters intersect on a limaçon.

Solution by the Rev. G. H. Hopkins, M.A.; S. Watson; *the* Proposer; *and others.*



Join P, the intersection of the two smaller circles, with B and C, the ends of their diameters; then, since BPO and OPC are right angles, BPC is a straight line. Let D, E, F be the centres of the circles; then DE and DF are perpendicular to BO and OC respectively; also EF is parallel to BC, and therefore perpendicular to OP. Now we have

$$\angle\,\text{BOP} = \text{DEF} = \text{COA},$$
$$\angle\,\text{BOA} = \text{BOC} - \text{COA},$$
$$\angle\,\text{POA} = \text{BOC} - 2\text{COA};$$
$$\text{OP} = \text{OA}\cos\text{BOA}\cos\text{BOP}$$
$$= \text{OA}\cos\tfrac{1}{2}(\text{BOC}+\text{POA})\cos\tfrac{1}{2}(\text{BOC}-\text{POA})$$
$$= \text{OD}(\cos\text{BOC} + \cos\text{POA}).$$

If, therefore, a circle OQD be drawn on OD as diameter, the locus of P will be obtained by taking on the *prolongation* of every such chord as OQ (or from Q *towards* O if BOC be an obtuse angle) a part QP equal to the constant length OD cos BOC; and this is the mode of generation of a limaçon.

3093. (Proposed by G. M. MINCHIN, B.A.)—A sphere is made to rotate about a diameter with a given angular velocity, and is then projected horizontally with a given velocity, in a plane perpendicular to the axis of rotation; show that the locus of the instantaneous axis is a parabolic cylinder.

Solution by MATTHEW COLLINS, B.A.

Let the sphere whose centre is O and radius a be projected with a velocity β along the horizontal line OX, and let ω be its angular velocity about a horizontal diameter perpendicular to OX. At the beginning of the motion, when $t=0$, let $\mathrm{OP}=r$ and $\mathrm{XOP}=\theta$ be the coordinates of a point P of the globe, then at the end of t seconds the rectangular coordinates of the same point P will plainly be

$$\mathrm{X}=\beta t+r\cos(\theta+\omega t) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

and Y (measured vertically downwards) $=\tfrac{1}{2}gt^2+r\sin(\theta+\omega t)$ ...... (2).

When the point P belongs to the instantaneous axis at the end of t seconds, the two foregoing equations differentiated give

$$\frac{d\mathrm{X}}{dt}=\beta-r\omega\sin(\theta+\omega t)=0,\quad \frac{d\mathrm{Y}}{dt}=gt+r\omega\cos(\theta+\omega t)=0 \;\ldots\ldots\ (3,4).$$

By eliminating $\cos(\theta+\omega t)$ from (1) and (4), and $\sin(\theta+\omega t)$ from (2) and (3), we find $\quad (g-\beta\omega)\,t-\mathrm{X}\omega=0,\quad (\tfrac{1}{2}gt^2-\mathrm{Y})\,\omega+\beta=0$ .... (5, 6).

Lastly, by eliminating t from (5) and (6), we find

$$\mathrm{X}^2=\left(\mathrm{Y}-\frac{\beta}{\omega}\right)\cdot\frac{(g-\beta\omega)^2}{\tfrac{1}{2}g\omega^2},$$

the equation of the required locus, which is therefore a parabolic cylinder when the projected body is any solid of revolution, revolving with an angular velocity ω about its axis of revolution, supposed horizontal.

[When $r=a$, we find from (3) and (4)

$$a^2\omega^2=\beta^2+g^2t^2,\quad\text{or}\quad t'=\frac{(a^2\omega^2-\beta^2)^{\frac{1}{2}}}{g};$$

hence there will be no instantaneous axis in the globe after t' seconds.

The Proposer's solution is given on p. 26 of this volume of the *Reprint*.]

3166. (Proposed by J. F. MOULTON, M.A.)—A plane is taken through a rectilinear generator of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, and the shortest distance between it and the next. Show that the equation of the parallel plane through the centre is

$$\frac{ax\,(b^2+c^2)}{\sin\theta}+\frac{by\,(c^2+a^2)}{\cos\theta}=cz\,(a^2-b^2),$$

and that such planes envelope the cone

$$\left\{ax\,(b^2+c^2)\right\}^{\frac{2}{3}}+\left\{by\,(c^2+a^2)\right\}^{\frac{2}{3}}=\left\{cz\,(a^2-b^2)\right\}^{\frac{2}{3}}.$$

Solution by Professor WOLSTENHOLME.

The equation of any generator of one system is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c},$$

and the direction cosines of the shortest distance between this generator and its consecutive will therefore be as $-\frac{\sin\theta}{a}:\frac{\cos\theta}{b}:\frac{1}{c}$; also the equation of any plane through the generator is

$$\mathrm{L}\left(\frac{x}{a}\cos\theta+\frac{y}{b}\right)\sin\theta-1+\mathrm{M}\left(\frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta-\frac{z}{c}\right)=0,$$

and the condition that this plane may contain the shortest distance is

$$\mathrm{L}\left(-\frac{\sin\theta\cos\theta}{a^2}+\frac{\sin\theta\cos\theta}{b^2}\right)+\mathrm{M}\left(-\frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{b^2}-\frac{1}{c^2}\right)=0,$$

or the equation required is

$$\left(\frac{x}{a}\cos\theta+\frac{y}{b}\sin^2\theta-1\right)\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}+\frac{1}{c^2}\right)$$
$$+\sin\theta\cos\theta\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{x}{a}\sin\theta-\frac{y}{b}\cos\theta-\frac{z}{c}\right)=0,$$

or $$\frac{x}{a}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\cos\theta+\frac{y}{b}\left(\frac{1}{a^2}+\frac{1}{c^2}\right)\sin\theta+\frac{z}{c}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\sin\theta\cos\theta$$
$$=\sin^2\theta\left(\frac{1}{a^2}+\frac{1}{c^2}\right)+\cos^2\theta\left(\frac{1}{b^2}+\frac{1}{c^2}\right);$$

and the equation of the parallel plane through the centre is the one given by Mr. MOULTON, which envelopes the cone specified at the end of the question.

[The envelope of the plane itself is one which has already been given two or three times in these volumes in the general form; viz., the envelope of

$$\mathrm{A}\cos 2\theta+\mathrm{B}\sin 2\theta+\mathrm{D}\cos\theta+\mathrm{E}\sin\theta+\mathrm{F}=0.]$$

3409. (Proposed by R. TUCKER, M.A.)—Two circles are drawn through the centre of the circumscribing circle of a triangle to touch the same two sides; prove that, if ρ_1, ρ_2 be the radii,

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{r}{\rho_1\rho_2}=\frac{r+2\mathrm{R}}{\mathrm{R}^2}.$$

Solution by S. WATSON; *the* PROPOSER; *and others.*

Suppose the circles to touch AB, AC; and let ρ be the radius of either,

then $$(\mathrm{R}\cos\mathrm{C}-\rho)^2+(\mathrm{R}\sin\mathrm{C}-\rho\cot\tfrac{1}{2}\mathrm{A})^2=\rho^2;$$

therefore $$\cot^2\tfrac{1}{2}\mathrm{A}.\rho^2-2\mathrm{R}(\cos\mathrm{C}+\sin\mathrm{C}\cot\tfrac{1}{2}\mathrm{A})\rho+\mathrm{R}^2=0,$$

and the two roots of this are the values of ρ_1, ρ_2; hence we have

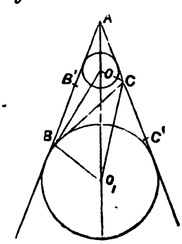
$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{2R(\cos C+\sin C\cot\frac{1}{2}A)}{R^2}=\frac{2R(1+2\operatorname{cosec}\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C)}{R^2}$$

$$=\frac{2R+r\operatorname{cosec}^2\frac{1}{2}A}{R^2};\quad\text{also}\quad\frac{r}{\rho_1\rho_2}=\frac{r\cot^2\frac{1}{2}A}{R^2}=\frac{r\operatorname{cosec}^2\frac{1}{2}A-r}{R^2};$$

hence, by subtraction, we get the required result.

3500. (Proposed by the Rev. G. H. HOPKINS, M.A.)—If O be the centre of the inscribed circle of the triangle ABC, and O_1 the centre of the circle escribed to the side BC; prove that $AO.AO_1 = AB.AC$.

Solutions (1) *by the* Rev. J. WHITE, M.A.; R. W. GENESE, B.A.; *and others*; (2) *by the* PROPOSER, *and* G. S. CARR, M.A.; (3) *by* J. J. SIDES.



1. O and O_1 must lie on the bisector of the angle A. Draw OB, O_1C which bisect respectively the angle B, and the external angle at C. Half the external angle at C is equal to half the sum of the internal angles at A and B, and therefore equal to the sum of the angles BAO and ABO. Hence the angles ACO_1, AOB are equal, being the supplements of equal angles. Consequently the triangles BOA and O_1CA (the angles at A being equal) are similar; therefore $AO.AO_1 = BA.AC$.

2. *Otherwise.*—The circle on OO_1 as diameter will cut AB, AC in points B′, C′, such that $AB' = AC$, and $AC' = AB$; therefore $AO.AO_1 = AB.AB' = AB.AC$.

3. *Or again;* $AO.AO_1 = rr_1\operatorname{cosec}^2\frac{1}{2}A = \frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{bc}{(s-b)(s-c)} = bc.$

3324. (Proposed by R. W. GENESE.)—AA′ is a diameter of a conic. If from points on any fixed chord through A′ tangents be drawn meeting the tangent at A in R, R′; prove that AR + AR′ is constant.

I. *Solution by* STEPHEN WATSON.

Take AA′ and the tangent at A for axes; then the equation of the ellipse, a diameter of which is AA′, and of a line through A′, are

$$\frac{y^2}{b^2}+\frac{(x-a)^2}{a^2}=1,\quad y=m(x-2a)\ \ldots\ldots\ldots\ldots\ldots\ldots\ (1,2).$$

Also the equations of tangents at the points where the lines $y = rx$, $y = r_1x$, meet (1), are

$$2a^2ry + (b^2 - a^2r^2)\,x = 2ab^2, \quad \text{and} \quad 2a^2r_1y + (b^2 - a^2r_1^2)\,x = 2ab^2,$$

and the condition that these shall meet on (1) is

$$a^2rr_1m + b^2(r + r_1) = 0, \quad \text{or} \quad \frac{b^2}{a^2}\left(\frac{1}{r} + \frac{1}{r_1}\right) = -m.$$

Hence putting $x = 0$, in the equations of the two tangents above, the sum of the resulting values of y is $\dfrac{b^2}{a}\left(\dfrac{1}{r} + \dfrac{1}{r_1}\right) = -am$, a constant so long as the line (2) is fixed, and being independent of b, holds good for all ellipses which have AA′ for a diameter; and also for all hyperbolas having the same diameter.

II. *Solution by the* Proposer.

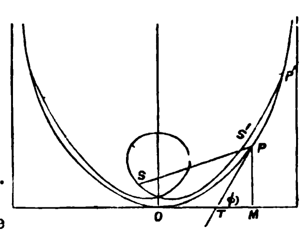
It is obvious that for one position of R there can only be one of R′; and *vice versâ*. There must therefore be a linear relation between the lengths AR $(=\alpha)$, AR′ $(=\alpha')$. Let it be $A\alpha\alpha' + B\alpha + C\alpha' + D = 0$; then, taking A′ as the point from which tangents are drawn, we see that α and α' are infinite together; hence $A = 0$.

From the interchangeability of α and α' we see that $B = C$; therefore $\alpha + \alpha' =$ a constant, and this constant is obviously the length AT cut off by the fixed chord from the tangent at A.

[If the fixed chord pass through A the sum of the *reciprocals* of AR, AR′ will be constant.]

3363. (Proposed by Professor Wolstenholme.) — In the (so-called) catenary of equal strength, $y = a \log \sec \dfrac{x}{a}$; if at each point P be described the equiangular spiral of closest contact and S be its pole, PS is of constant length, and the arc described by S between any two positions is equal to the corresponding arc of the curve.

Solution by the Proposer.



Let $OM = x$, $MP = y$, then

$$\frac{dy}{dx} = \tan\frac{x}{a} \equiv \tan\phi;$$

therefore $x = a\phi$, $y = a \log \sec \phi$,

and $$\frac{ds}{d\phi} = a \sec\phi \equiv \rho,$$

also $\rho\dfrac{d\rho}{ds} = a \sec\phi \tan\phi$ or $\dfrac{d\rho}{ds} = \tan\phi$.

But in the equiangular spiral, if β be the angle of the spiral, $\dfrac{d\rho}{ds} = \cot\beta$, whence, if S be the pole of the spiral of

closest contact at P,

$$\angle TPS = \tfrac{1}{2}\pi - \phi = \angle TPM, \quad \text{also} \quad PS = \rho \sin \beta = \rho \cos \phi = a;$$

or if X, Y be coordinates of S,

$$X = a\phi - a \sin 2\phi, \quad Y = a \log \sec \phi + a \cos 2\phi;$$

and if σ be the arc described by S,

$$\left(\frac{d\sigma}{d\phi}\right)^2 = a^2 \left\{(1-2 \cos 2\phi)^2 + (\tan \phi - 2 \sin 2\phi)^2\right\} = a^2 \sec^2 \phi, \text{ or } \frac{d\sigma}{d\phi} = \frac{ds}{d\phi}.$$

Hence the arcs described by S, P are equal, and the curves are as in the figure, PS soon tending to become vertical. The curvature at S of the locus of S is three times the curvature at P, since the tangent at S makes with the axis of x an angle 3ϕ.

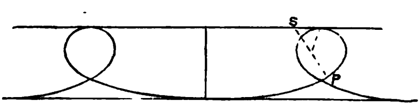
It may readily be shown that, in any curve, if S be the pole of the equiangular spiral of closest contact at P, $PS = \rho \left\{1 + \left(\frac{d\rho}{ds}\right)^2\right\}^{-\frac{1}{2}}$;

and if σ be the arc described by S, that $\frac{d\sigma}{ds} = \pm \rho \frac{d^2\rho}{ds^2} \div \left\{1 + \left(\frac{d\rho}{ds}\right)^2\right\}$.

Hence, in any curve whose intrinsic equation is $\frac{ds}{d\phi} = a (\cos n\phi)^{\frac{1}{n}}$, we shall have $\frac{d\sigma}{ds} = n$, and the curvature at S of the locus of S is $\frac{2n-1}{n}$ times the curvature at P. Thus, in a cycloid, $n=1$, and the locus of S is an equal cycloid on the opposite side of the same base.

In the curve $s = a(\phi + \sin \phi)$, $n = \frac{1}{2}$, and the locus of S is a straight line.

This curve is somewhat as figured, consisting of an infinite number of similar and equal branches, and the locus of S is the upper straight line.

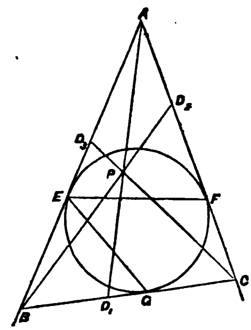


The radius of curvature at all the points where the curve touches the straight line is zero, the points being similar to the vertex of the curve $y^4 = ax^3$.

The nodes are where the tangent makes an angle ϕ with the lower line given by the equation $2 \sin \phi + \sin \phi \cos \phi = \pi - \phi$, and ϕ is therefore nearly $= \frac{1}{3}\pi$, so that the loops are drawn considerably too large in the figure.

3299. (Proposed by R. Tucker, M.A.)—A fixed circle touches two given straight lines AB, AC at the points E, F, the third line BC being always a tangent to the circle. Prove that the locus of the points in which the lines joining the angles with the points of contact of the circles escribed to the triangle ABC intersect, is a hyperbola having AB, AC for asymptotes.

I. *Solution by* Stephen Watson.



Let EF $= a$, AE $=$ AF $= b$; denote EF, FA, AE by the trilinear equations $\alpha = 0$, $\beta = 0$, $\gamma = 0$; and let D_1, D_2, D_3 be the points of contact of the escribed circles of the triangle ABC with BC, CA, AB. The trilinear equation of the circle touching AB, AC at E and F is $\alpha^2 = \beta\gamma$, and if the equation of the line EG be $\alpha + r\gamma = 0$, that of the tangent BC at G is

$$2r\alpha + \beta + r^2\gamma = 0 \quad \ldots\ldots\ldots\ldots (1).$$

In (1) put $\beta = 0$, then $2\alpha + r\gamma = 0$, and this combined with $a\alpha + b\gamma = \Delta$, gives $\alpha = -\dfrac{2r\Delta}{2b - ar}$;

therefore $\text{FC} = \dfrac{2r\Delta}{(2b-ar)\sin \text{F}} = \text{AD}_2$;

hence, if $(\alpha', \beta', \gamma')$ denote the point D_2,

$$\alpha' = (b - \text{AD}_2)\sin \text{F} = \frac{4\Delta(b-ar)}{a(2b-ar)},\quad \beta' = 0,\quad \gamma' = \text{AD}_2 \sin \text{A} = \frac{2ra\Delta}{b(2b-ar)};$$

therefore the equations of BD_2, CD_3 are, respectively,

$$2a^2r\alpha + a^2\beta + 4b(ar-b)\gamma = 0,\quad 2a^2\alpha + 4b(a-br)\beta + a^2r\gamma = 0 \ldots\ldots (2, 3),$$

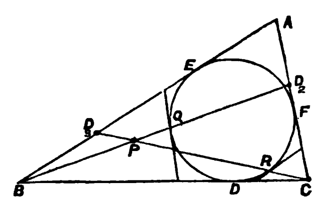
and it is well known that BD_2, CD_3, AD_1 are concurrent.

Eliminating r from (2) and (3), the result gives

$$4a^2\left\{a\alpha + b(\beta+\gamma)\right\}^2 = (4b^2 - a^2)^2\beta\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and this conic touches AB, AC at the points where the lines $a\alpha + b\beta = 0$ and $a\alpha + b\gamma = 0$, respectively meet them; but those lines are parallel to AB, AC, and therefore meet them at infinity; hence (4) is a hyperbola having AB, AC for asymptotes.

II. *Solution by* R. W. Genese, B.A.



Since B is the centre of similitude of the circle DEF and the circle touching AC in D_2, if BD_2 meet the the circle in Q, the tangent at Q will be parallel to AC; therefore Q is a fixed point. Similarly CD_3 meets the circle in a fixed point R.

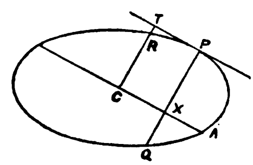
Now take four positions of the tangent BC; then, if (B) stand for the anharmonic range $(\text{B}_1\text{B}_2\text{B}_3\text{B}_4)$, we have (B) $=$ (C), therefore Q (B) $=$ R (C), *i. e.*, Q (P) $=$ R (P).

Thus a conic will pass through Q, R, and the four positions of P. Fixing three of these positions, we see that any other will lie on a fixed conic through Q and R.

Taking BC parallel to either AB or AC, we see that the points at infinity on the locus of P are in the direction of AB or AC.

3435. (Proposed by Professor Wolstenholme.)—If PQ be a chord of an ellipse meeting the circle of curvature at P in Q′, prove that $PQ' : PQ = d^2 : d'^2$, where d, d' are the diameters parallel to the tangent at P, and to the chord, respectively.

Solution by R. W. Genese, B.A.



Let CA and CR, the semi-diameters parallel to the tangent at P and to PQ respectively, meet PQ and the tangent in X and T. Then we have

$$PX \cdot XQ : CA^2 - CX^2 = CR^2 : CA^2,$$

$$TC^2 - CR^2 : TP^2 = CR^2 : CA^2;$$

therefore, since $TC = PX$, and $TP = CX$,

$$PX \cdot XQ + PX^2 - CR^2 : CA^2 = CR^2 : CA^2;$$

therefore $\quad PX \cdot PQ - CR^2 = CR^2$ or $PX \cdot PQ = 2CR^2$.

But $PX \cdot PQ' = 2CA^2$, whence the theorem in question.

[This theorem may also be deduced from the fact that $PX \cdot PQ : CR^2$ is unaltered by projection. In a circle, $PX \cdot PQ$ = a constant $= 2PC^2 = 2CR^2$.]

3350. (Proposed by R. W. Genese.)—M is any point on the tangent at a vertex A of an ellipse; on AC take AO equal to the semi-minor axis, and along AM take MT, MT′ each equal to OM; then the tangents from T and T′ will be parallel to CM.

Solution by Stephen Watson.

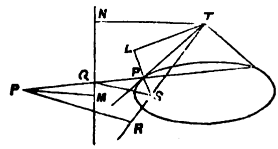
Let (x', y') be the point of contact of either tangent drawn parallel to CM; then the equation of the tangent is $\dfrac{x'x}{a^2} + \dfrac{y'y}{b^2} = 1$ .... (1),

and if $AM = z$, we have $-\dfrac{b^2x'}{a^2y'} = \dfrac{z}{a}$, and $\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} = 1$.

Putting the values of x', y' derived from these two last equations in (1), and then making $x = a$, the resulting values of y are $y = z \pm (b^2+z^2)^{\frac{1}{2}}$; hence the tangents parallel to CM pass through T and T′.

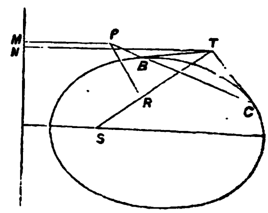
3301. (Proposed by the Rev. A. F. Torry, M.A.)—If from a point P on the polar of T, with respect to a conic whose focus is S, PR be drawn at right angles to ST, and PM, TN be the perpendiculars upon the corresponding directrix, prove that $SR \cdot ST = e^2 \cdot PM \cdot TN$.

I. *Solution by* Professor WOLSTENHOLME.



Let the polar of T meet the directrix in Q, then QS is perpendicular to ST or parallel to PR; hence SR to PM is a constant ratio so long as T is constant, and taking P on the ellipse, we see that this ratio $= e \cos \text{PST}$; but if TL be drawn perpendicular to SP, SL : TN $= e : 1$; therefore ST cos PST : TN $= e : 1$; therefore SR . ST $= e^2$. PM . TN for the point on the ellipse, and therefore for any point on the polar of T.

II. *Solution by the* Rev. J. L. KITCHIN, M.A.



The coordinates of S, T may be taken $(-ae, 0)$, (h, k); hence the equation to ST is

$$(ae+h)\, y = (ae+x)\, k.$$

Take (X, Y) as the coordinates of P; then

$$\text{PM} = \text{X} + \frac{a}{e}, \quad \text{TN} = h + \frac{a}{e};$$

therefore $e^2.\text{PM}.\text{TN} = (a+e\text{X})(a+eh)\ldots$ (1).
Now the equation to PR is

$$y - \text{Y} = -\frac{ae+h}{k}(x - \text{X});$$

therefore the length of the perpendicular from $(-ae, 0)$ on it is

$$\text{SR} = \frac{k\text{Y} + (ae+h)(ae+\text{X})}{\text{ST}},$$

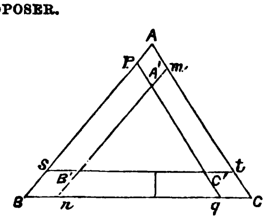
therefore $\quad \text{SR}.\text{ST} = ky + (ae+h)(ae+\text{X}),$

but the equation to BC is $k\text{Y} + h\text{X}(1-e^2) = a^2(1-e^2)$;

therefore $\quad \text{SR}.\text{ST} = (a^2 - h\text{X})(1-e^2) + (ae+h)(ae+\text{X})$
$\qquad\qquad = (a+eh)(a+e\text{X}) = e^2.\text{PM}.\text{TN}$, by (1).

3451. (Proposed by A. MARTIN.)—Show that the average area of all the circles that can be drawn in a given triangle is one-tenth of the inscribed circle.

I. *Solution by the* PROPOSER.



Draw mn, pq, st parallel to the sides AB, AC, BC respectively of the given triangle ABC, and at the distance x from them. Then a circle whose radius is x may have its centre anywhere within the triangle A′B′C′; hence the area of that triangle may be taken as the measure of the number of such circles.

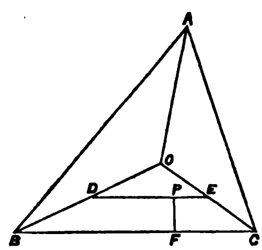
Now the triangle A′B′C′ is similar to ABC, and the radius of its inscribed circle

is $r-x$; therefore its area is $\frac{(r-x)^2}{r^2}\Delta$, where Δ is the area of ABC.

The area of the circle whose radius is x is πx^2; therefore the average area required is

$$\frac{\int_0^r \pi x^2 \frac{\Delta}{r^2}(r-x)^2 dx}{\int_0^r \frac{\Delta}{r^2}(r-x)^2 dx} = \frac{3\pi}{r^3}\int_0^r x^2 (r-x)^2 dx = \frac{\pi r^2}{10}.$$

II. *Solution by* STEPHEN WATSON.



Let O be the inscribed centre of the given triangle ABC; join OA, OB, OC; draw DE parallel to BC, and from any point P in DE draw PF perpendicular to BC. Put PF $=\rho$, DE$=z$; then $r : a = r-\rho : z$, therefore $\rho = \frac{r(a-z)}{a}$; hence the sum of the areas of all the circles that can be drawn within the triangle, having P for centre, is

$$\int_0^\rho \pi x^2 dx = \tfrac{1}{3}\pi\rho^3,$$

and when P takes all positions in the triangle OBC, the sum is

$$\tfrac{1}{3}\pi\int_0^r \rho^3 z\,d\rho = -\frac{r^4\pi}{3a^4}\int_a^0 (a-z)^3 z\,dz = \tfrac{1}{60}\pi r^4 a.$$

For the same positions of P, the number of circles formed is

$$\int_0^r \rho z\,d\rho = \tfrac{1}{6}r^2 a;$$

hence the average required is obviously $\frac{\frac{1}{60}\pi r^4(a+b+c)}{\frac{1}{6}r^2(a+b+c)} = \frac{\pi r^2}{10}$.

3507. (Proposed by Professor CAYLEY.)—Show that, for the quadric cones which pass through six given points, the locus of the vertices is a quartic surface having upon it twenty-five right lines; and, thence or otherwise, that for the quadric cones passing through seven given points the locus of the vertices is a sextic curve.

Solution by Professor WOLSTENHOLME.

If $P=0$, $Q=0$, $R=0$, $S=0$ be four quadrics through the six points, $U \equiv aP+bQ+cR+dS=0$ is the general equation of a quadric through them. If this be a cone, and (x, y, z, w) its vertex, we shall have the

equations $\qquad \dfrac{dU}{dx}=0,\ \dfrac{dU}{dy}=0,\ \dfrac{dU}{dz}=0,\ \dfrac{dU}{dw}=0;$

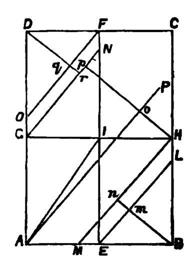
whence the locus of the vertex is the quartic surface

$$\begin{vmatrix} \dfrac{dP}{dx}, & \dfrac{dP}{dy}, & \dfrac{dP}{dz}, & \dfrac{dP}{dw} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{dR}{dx}, & \dfrac{dR}{dy}, & \dfrac{dR}{dz}, & \dfrac{dR}{dw} \end{vmatrix} = 0.$$

This locus contains the fifteen straight lines joining the six points two and two, since with any point on such line as vertex we can describe a quadric cone containing the six points. It also, for a similar reason, contains the ten lines of intersection of pairs of planes which can be drawn through the six points, making twenty-five in all. If then we have seven points, and take the two quartic surfaces corresponding to two sets of six, the degree of their curve of intersection is 16; but part of this intersection is the ten straight lines joining the five points common to the two sets of six, which do not apply, since they do not lie on all the quartics. We shall have, as the remaining intersection and the true locus of the vertex of all cones through the seven points, a sextic curve.

3440. (Proposed by S. WATSON.)—A line is drawn at random across a window containing four equal rectangular panes; find the respective chances of its crossing one, two, or three of the panes.

Solution by the PROPOSER.



Let ABCD represent the window, and let AP be parallel to any random line crossing the window. Draw EL, HM, GN, FO parallel to AP, and Bmn, Hop, Dqr perpendicular to AP, meeting the several lines as in the diagram; also join AI. Put AB $=2a$, BC $=2b$, AI $=c$, $\angle$ IAB $=\alpha$, and PAB $=\theta$. Then when $\theta<\alpha$, the numbers of lines that can be drawn parallel to AP to cross one, two, or three of the panes, are respectively

$$\text{B}m+\text{D}q = 2\text{B}m = 2a\sin\theta \quad\text{............ (1)},$$

$$mn+qr = 2mn = 2(b\cos\theta - a\sin\theta) \quad\text{...... (2)},$$

$$\text{H}p = 2a\sin\theta \quad\text{...................... (3)}.$$

Similarly when $\theta>\alpha$, the numbers for crossing one (or three), and two, of the panes are $\qquad 2b\cos\theta,\ 2(a\sin\theta - b\cos\theta)$ .... (4, 5).

Now all necessary values of θ are comprised between $\theta=0$ and $\frac{1}{2}\pi$; hence the whole number of lines drawn is expressed by

$$\int_0^{\frac{1}{2}\pi} 2(a\sin\theta + b\cos\theta)\,d\theta = 2(a+b),$$

and the respective chances of crossing one (or three), and two, of the

panes are $$\frac{1}{2(a+b)}\left\{\int_0^{\alpha}(1)\,d\theta+\int_{\alpha}^{\frac{1}{2}\pi}(4)\,d\theta\right\}=1-\frac{c}{a+b},$$

$$\frac{1}{2(a+b)}\left\{\int_0^{\alpha}(2)\,d\theta+\int_{\alpha}^{\frac{1}{2}\pi}(5)\right\}=\frac{2c}{a+b}-1.$$

[This solution is in conformity with Mr. CROFTON's view of random lines in a plane, which was first published in a *Note on Local Probability*, on pp. 84—86 of Vol. VII. of the *Reprint*, and afterwards formed the basis of a very interesting and valuable paper, read before the Royal Society, and printed in their *Transactions*. On the supposition that a random line is a line drawn at random through a point taken at random on the window, another solution has been given by Mr. McCOLL, on p. 31 of this volume of the *Reprint*.]

3524. (Proposed by DISCIPULUS.)—A boy, on being asked what $\frac{12}{13}$ of a certain fraction was, made a mistake common enough with beginners; he divided the fraction by $\frac{12}{13}$, and so got an answer which exceeded the correct one by $\frac{25}{162}$. Required the correct answer.

Solution by ISABELLA M. WARD; J. LANE; H. MURPHY; *and others.*

Let us suppose that x denotes the fraction of which the boy was to find $\frac{12}{13}$; then $\frac{12}{13}x$ is the correct answer, and $\frac{13}{12}x$ is the boy's answer. Hence we have $$\frac{13}{12}x-\frac{12}{13}x=\frac{25}{162},\ \text{whence}\ x=\frac{26}{27}.$$
The correct answer is therefore $\frac{12}{13}$ of $\frac{26}{27}$, or $\frac{4}{9}$.

3442. (From WHITWORTH's *Choice and Chance*.)—A bag contains n tickets numbered $1, 2, 3, \ldots n$. A man draws (1) two tickets at once, and is to receive a number of sovereigns equal to the product of the numbers drawn; find his expectation; also find (2) the value of the expectation if three tickets were drawn and their continued product taken.

Solution by MILLICENT COLQUHOUN.

1. There are $\frac{1}{2}n(n-1)$ possible events, all equally probable; and their total value $$=\tfrac{1}{2}\left\{(1+2+\ldots+n)^2-(1^2+2^2+\ldots+n^2)\right\}$$
$$=\tfrac{1}{8}n^2(n+1)^2-\tfrac{1}{12}n(n+1)(2n+1)=\tfrac{1}{24}n(n+1)(3n+2)(n-1).$$
The expectation is therefore $\frac{1}{12}(n+1)(3n+2)$ £.

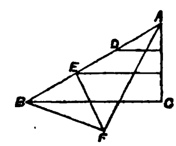
2. There are $\frac{1}{6}n(n-1)(n-2)$ equally probable events, and their total value
$$=\tfrac{1}{6}\left\{(1+2+\ldots+n)^3-3(1+\ldots+n)(1^2+\ldots+n^2)+2(1^3+2^3+\ldots+n^3)\right\}$$
$$=\tfrac{1}{6}\left[\left\{\tfrac{1}{2}n(n+1)\right\}^3-3\left\{\tfrac{1}{2}n(n+1)\,.\,\tfrac{1}{6}n(n+1)(2n+1)\right\}+2\left\{\tfrac{1}{2}n(n+1)\right\}\right].$$

Hence the value of the expectation is

$$\frac{n^2(n+1)^2}{n(n-1)(n-2)}\left\{\tfrac{1}{8}n(n+1)-\tfrac{1}{4}(2n+1)+\tfrac{1}{2}\right\} = \tfrac{1}{8}n(n+1)^2 \text{ £}.$$

3418. (Proposed by J. B. SANDERS.)—Determine that portion of an inclined plane, equal to its height, which a body, in falling down the plane, passes over in the same time it would fall freely through the height.

Solution by A. MARTIN; T. MITCHESON, F.E.I.S.; *and others.*



Put $AB = a$, $\angle ABC = \beta$, and, DE being the portion of the plane required, let $AE = x$, and $AD = y$; then we have $\quad x - y = a \sin\beta$ .... (1).

Now the respective times down AC, AD, AE are

$$\left(\frac{2a\sin\beta}{g}\right)^{\frac{1}{2}},\quad \left(\frac{2y}{g\sin\beta}\right)^{\frac{1}{2}},\quad \left(\frac{2x}{g\sin\beta}\right)^{\frac{1}{2}};$$

therefore $$\left(\frac{2x}{g\sin\beta}\right)^{\frac{1}{2}} - \left(\frac{2y}{g\sin\beta}\right)^{\frac{1}{2}} = \left(\frac{2a\sin\beta}{g}\right)^{\frac{1}{2}};$$

or $$x^{\frac{1}{2}} - y^{\frac{1}{2}} = a^{\frac{1}{2}}\sin\beta \quad \text{.............................. (2).}$$

From (1) and (2) we find

$$x = \tfrac{1}{4}a(1+\sin\beta)^2,\quad y = \tfrac{1}{4}a(1-\sin\beta)^2.$$

[Mr. MARTIN's results may be put in a simpler form, namely, $AE = AB\cos^2\frac{1}{2}A$, $AD = AB\sin^2\frac{1}{2}A$. Hence the point E may be found by drawing BF perpendicular to the bisector of the angle A, and then from F drawing FE perpendicular to AB. We have, also, $AD = BE$.]

3342. (Proposed by H. McCOLL.)—A point is taken at random inside an equilateral triangle, and from it a perpendicular is drawn to each of the sides. Show that $3\log 2 - 2$ is the chance that straight lines equal respectively to these three perpendiculars can be the sides of an acute-angled triangle.

Solution by the PROPOSER.

Let ABC be the equilateral triangle. In the side AC take two random points E and F. From the point nearest to A of these two draw a straight line parallel to AB, and from the point nearest to C draw a straight line parallel to CB; and let the lines so drawn intersect at P. Then P may be considered a random point in the triangle; for if the triangle be divided into any number of equal parts, all the parts will stand the same chance of containing P.

Now the perpendiculars from P upon the sides of the triangle are pro-

portional to the three random segments of the side AC. The chance, therefore, of the three perpendiculars forming an acute-angled triangle is equal to the chance of these three segments forming an acute-angled triangle. In other words, the above question is virtually the same as the EDITOR's Question 1100, of which solutions have already been given in the *Reprint*, Vol. II. p. 74, Vol. V. p. 33, and Vol. XV. p. 35.

3502. (Proposed by R. W. GENESE, B.A.)—Given the general equation to the two tangents from a point $(a, 0)$ to a conic, viz.,

$$y^2(a\alpha^2+2b\alpha+c)+2y(x-\alpha)(b'\alpha+c')+c''(x-\alpha^2)=0,$$

find the equation to the polar of the point.

Solution by MATTHEW COLLINS, B.A.

The given equation arranged according to the powers of α is

$$\alpha^2(ay^2-2b'y+c'')+2\alpha(by^2+b'xy-c'y-c''x)+cy^2+2c'xy+c''x^2=0,$$

and when α varies the equation of the envelope is (SALMON's *Conics*, Art. 283)

$$(by^2+b'xy-c'y-c''x)^2=(ay^2-2b'y+c'')(cy^2+2c'xy+c''x^2),$$

and as the terms containing x^2, xy, and x^2y cancel out, the remaining terms divided by y^2 give

$$(b^2-ac)y^2+2xy(bb'-ac')+x^2(b'^2-ac'')$$
$$+2y(b'c-bc')+2x(b'c'-bc'')+c'^2-cc''=0,$$

for the equation of the enveloped conic. Hence (SALMON's *Conics*, Art. 144) the required equation of the polar of the point (x', y') is

$$(b^2-ac)yy'+(xy'+x'y)(bb'-ac')+xx'(b'^2-ac'')+(y+y')(b'c-bc')$$
$$+(x+x')(b'c'-bc'')+c'^2-cc''=0.$$

When $x'=\alpha$ and $y'=0$, this becomes

$$y(bb'-ac')+x(b'^2-ac'')+b'c'-bc''+\frac{y(b'c-bc')+x(b'c'-bc'')+c'^2-cc'}{\alpha}=0;$$

and when $\alpha=\infty$, the equation of the polar is

$$y(bb'-ac')+x(b'^2-ac'')+b'c'-bc''=0.$$

3404. (Proposed by the Rev. R. TOWNSEND, M.A.)—The transverse section of a flexible cord, in free equilibrium under the action of a central force, being supposed to vary as the length of the perpendicular from the centre of force on the tangent to the curve of equilibrium; show that the law of force is the same (with its sign changed) as for a material point describing freely the same curve under the action of a force emanating from the same centre.

I. *Solution by* WILLIAM ROBERTS, Jun.

When the form of F is known, the equation to the catenary is

$$p = \frac{C}{\int Fmdr}, \quad \text{or} \quad F = \frac{-C}{mp^2} \cdot \frac{dp}{dr};$$

and if m varies as p, we have $F = -\frac{C}{p^3} \cdot \frac{dp}{dr}$.

II. *Solution by the* PROPOSER.

For since, by well known principles, in the former case,

$$Fdr = \mp \frac{1}{e} dT, \; = \mp k^2 . \frac{1}{p} d\frac{1}{p}, \; = \mp \tfrac{1}{2} k^2 . d\left(\frac{1}{p^2}\right);$$

and, in the latter case,

$$Fdr = \pm vdv, \; = \pm h^2 . \frac{1}{p} d\frac{1}{p} = \pm \tfrac{1}{2} h^2 . d\left(\frac{1}{p^2}\right);$$

therefore, &c.

III. *Solution by* G. S. CARR, M.A.

Let mf be the repulsive force acting on an element δs of the cord, m being the mass of the element, and T the tension at that point. Then (by TODHUNTER'S *Statics*, Art. 192)

$$\frac{dT}{ds} = -mf\frac{dr}{ds}; \text{ therefore } \frac{dT}{dr} = -mf.$$

Also if p = perpendicular from centre on tangent, $T = \frac{C}{p}$,

therefore $\quad -mf = \frac{d(Cp^{-1})}{dr}$; therefore $mf = \frac{C}{p^2} \cdot \frac{dp}{dr}$ .... (1).

Now m varies as the section of the cord, which $\propto p$ by hypothesis; therefore $m \propto p$.

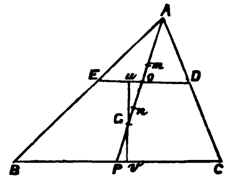
Let $m = kp$. Substitute in (1); therefore

$$f = \frac{C}{k} \frac{1}{p^3} \frac{dp}{dr},$$

the differential equation to the orbit described by a particle under the action of an attractive central force f. (See TAIT'S *Dynamics*, Art. 130.)

3497. (Proposed by T. MITCHESON, F.E.I.S.)—Four men support a quadrilateral slab (formed by that portion of a triangle cut off by a line parallel to the base, and at the distance of one-pth of the height of the triangle from the vertex), one at each corner. Show that the weight borne by the two men at the shorter side is to that borne by the other two as $p+2 : 2p+1$.

Solution by S. Watson; A. B. Evans, M.A.; *the* Proposer; *and others.*



Let BCDE be the slab; m, n, G the centres of gravity of the triangles ADE, ABC, and the slab, which must all lie in the line AP bisecting BC; w_1, w_2, w_3, w_4 the weights borne at D, E, B, C respectively; u the centre of gravity of w_1, w_2; v that of w_3, w_4; also let AP meet DE in o. Then uv must pass through G, and

$$\frac{w_1+w_2}{w_3+w_4} = \frac{\text{G}v}{\text{G}u} = \frac{\text{GP}}{\text{G}o} = \frac{\frac{1}{3}\text{AP}-n\text{G}}{(\frac{2}{3}-p^{-1})\,\text{AP}+n\text{G}} \;\ldots(1).$$

Again $$\frac{n\text{G}}{nm} = \frac{\Delta\,\text{ADE}}{\Delta\,\text{ABC}-\Delta\,\text{ADE}} = \frac{1}{p^2-1},$$

therefore $$n\text{G} = \frac{\text{A}n-\text{A}m}{p^2-1} = \frac{\frac{2}{3}(1-p^{-1})\,\text{AP}}{p^2-1} = \frac{2}{3}\cdot\frac{\text{AP}}{p\,(p+1)},$$

hence (1) becomes $$\frac{w_1+w_2}{w_3+w_4} = \frac{p\,(p+1)-2}{(2p-3)\,(p+1)+2} = \frac{p+2}{2p+1}.$$

3512. (Proposed by W. S. McCay, M.A.)—If a conic be cut harmonically by another, the following relation exists between their invariants:

$$2\Theta^3 - 9\Theta\Theta'\Delta + 27\Delta^2\Delta' = 0.$$

Solution by Professor Wolstenholme.

If the equations of the two conics be $x^2+y^2+z^2=0$, $ax^2+by^2+cz^2=0$, the four common points subtend, at any point on the former, the pencil $\dfrac{a-b}{a-c}$ (Wolstenholme's *Book of Mathematical Problems*, 822). This will be harmonic if $2a=b+c$, &c.; the required condition is

$$(b+c-2a)\,(c+a-2b)\,(a+b-2c) = 0,$$

or $$-2\,(a+b+c)^3 + 9\,(a+b+c)\,(bc+ca+ab) - 27abc = 0,$$

or $$2\Theta^3 - 9\Theta\Theta'\Delta + 27\Delta^2\Delta' = 0,$$

since $$\Delta = 1,\quad \Theta = a+b+c,\quad \Theta' = bc+ca+ab,\quad \Delta' = abc.$$

This equation, being truly homogeneous, is the condition required.

I may mention, perhaps, that a relation between these invariants expressing a projective property must be homogeneous when Δ, Θ, Θ', Δ' are (1) of 0, 1, 2, 3 dimensions, (2) of 3, 2, 1, 0 dimensions, and (3) all of the same dimensions. Such are $\Theta\Theta' = \Delta\Delta'$, $\Theta^2 = 4\Delta\Theta'$, ...

[This condition has also been given by Mr. Walker, in No. 38 of the *Quarterly Journal of Mathematics*.]

3444. (Proposed by W. Hogg, M.A.)—A particle is placed at a given distance from a uniform thin plate of indefinite extent, every particle of which attracts with a force varying inversely as the square of the distance; to find the time in which the particle will arrive at the surface of the plate.

Solution by G. S. CARR.

Let A be the particle; AB ($=a$) a perpendicular upon the plane; C a point in the plane at a distance BC ($=r$) from B; and let f be the attraction of a unit of area upon the particle at a unit of distance.

The resolved part, along AB, of the attraction of an elemental ring, of radius r, will be $f.2\pi ra\,(a^2+r^2)^{-\frac{3}{2}}\,dr$; hence the total resultant force perpendicular to the plane is

$$\int_0^\infty f\cdot\frac{2\pi ar}{(a^2+r^2)^{\frac{3}{2}}}\,dr = 2\pi f, \text{ a constant};$$

therefore the time of reaching the plane is $\left(\frac{a}{\pi f}\right)$.

3482. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Eliminate θ between the equations $\quad m = \dfrac{a\sin\theta\cos^2\theta}{(a\cos\theta+b)^2},\quad n = \dfrac{a\cos^3\theta+b}{(a\cos\theta+b)^2}.$

Solution by the Rev. W. H. LAVERTY, M.A.

To make the equations homogeneous, let $m = \dfrac{1}{a}$, $n = \dfrac{1}{\beta}$; also let $\dfrac{1}{a^2}+\dfrac{1}{\beta^2} = \dfrac{1}{\gamma^2}$, and $\cos\theta = x$; then

$$\frac{1}{\beta} = \frac{ax^3+b}{(ax+b)^2},\quad \frac{1}{\gamma^2} = \frac{a^2x^4+2abx^3+b^2}{(ax+b)^4};$$

therefore we have to eliminate x between

$$x^4(a^2-\gamma^2)\,a^2+x^3(2a^2-\gamma^2)\,2ab+x^2(6a^2b^2)+x\,(4ab^3)+b^2(b^2-\gamma^2) = 0,$$

$$x^3(a\beta)-x^2(a^2)-x\,(2ab)+b\,(\beta-b) = 0.$$

The result is therefore obtained by equating to zero the determinant

$$\begin{vmatrix} a(a^2-\gamma^2) & 2b(2a^2-\gamma^2) & 6ab^2 & 4ab^3 & b^2(b^2-\gamma^2) & . & . \\ . & a(a^2-\gamma^2) & 2b(2a^2-\gamma^2) & 6a^2b^2 & 4ab^3 & b(b^2-\gamma^2) & . \\ . & . & a(a^2-\gamma^2) & 2ab(2a^2-\gamma^2) & 6a^2b^2 & 4ab^2 & b(b^2-\gamma^2) \\ \beta & -a & -2b & b(\beta-b) & . & . & . \\ . & \beta & -a & -2ab & b(\beta-b) & . & . \\ . & . & \beta & -a^2 & -2ab & (\beta-b) & . \\ . & . & . & a\beta & -a^2 & -2a & (\beta-b) \end{vmatrix}$$

2685. (Proposed by J. J. WALKER, M.A.)—It is required to determine the conditions that the values of $\phi(x)$ may be real and have given signs, when the roots of $f(x) = 0$ are substituted for x, f and ϕ being rational

and integral functions; also to apply the general theory to the case of the values of $x^2+2px+q$ being real and positive when the roots of $x^2+2ax+b=0$ are substituted for x.

Solution by the PROPOSER.

Let a be a root of $f(x)=0$, then must $\phi(a)-z=0$, with the condition that z have a given sign, that is, $f(x)=0$ and $\phi(x)-z=0$ must have common roots, with the above condition. Eliminating x between these equations, let the result be $\psi(z)=0$, then the conditions among the coefficients of $f(x)$ and $\phi(x)$ which give all the roots of this equation real, a certain number of one sign, the rest of the other sign, are those required. To apply this general theory to the case proposed:

Eliminating x between $x^2+2ax+b=0$ and $x^2+2px+q-z=0$, the result, arranged by powers of z, is

$$z^2-2\left\{2a(a-p)-(b-q)\right\}z+4(a-p)(aq-bp)+(b-q)^2=0.$$

The condition that the roots of this equation should be real is

$$\left\{2a(a-p)-(b-q)\right\}^2-\left\{4(a-p)(aq-bp)+(b-q)^2\right\} \not< 0 \;\ldots\ldots\; (1),$$

which reduces to $(a-p)^2(a^2-b) \not< 0$, that is, either $a=p$ or $a^2-b \not< 0$.

The conditions that the roots, being both real, should be both positive, are $2a(a-p)-(b-q)>0$ and $4(a-p)(aq-bp)+(b-q)^2>0$ ... (2, 3).

If $a=p$, conditions (2) and (3) become $q-b>0$, $(q-b)^2>0$.

If $a^2=b$, condition (2) becomes $a^2-2pa+q>0$, as it evidently should, and (3) becomes $(a^2-2pa+b)^2>0$.

3462. (Proposed by J. J. WALKER, M.A.)—1. Prove the identity

$$\Sigma l(m-n)^2 . \Sigma(2m+2n-l)(m-n)^2-\Sigma mn\left\{\Sigma(m+n)^2\right\}^2$$

$$=9(m-n)^2(n-l)^2(l-m)^2,\quad l,\, m,\, n \text{ being any three quantities.}$$

2. The standard form for the discriminant of the binary quartic $(a\,b\,c\,d\,e \between x\,y)^4$, viz. $\Delta=\mathrm{I}^3-27\mathrm{J}^2$, may be transformed into $\Delta=\mathrm{IK}-3\mathrm{J}'^2$, where $\mathrm{J}'=3\mathrm{J}+c\mathrm{I}$; prove that this is equivalent to the above identity, if $\alpha,\beta,\gamma,\delta$ be the roots of the quartic, and $\alpha\beta+\gamma\delta=l$, $\alpha\gamma+\beta\delta=m$, $\alpha\delta+\beta\gamma=n$.

Solution by the PROPOSER.

1. Let $m-n=\lambda$, $n-l=\mu$, $l-m=\nu$; then the coefficient of λ^4 in the development of the left side will be $lm+ln-mn-l^2$, which is equal to $\mu\nu$. Consequently the aggregate of the terms λ^4, μ^4, ν^4 will be equal to $\lambda\mu\nu(\lambda^3+\mu^3+\nu^3)$ or $3\lambda^2\mu^2\nu^2$ (since $\lambda+\mu+\nu\equiv 0$, and therefore $\lambda^3+\mu^3+\nu^3=3\lambda\mu\nu$). Again, the coefficient of $\mu^2\nu^2$ is $2m^2-4mn+2n^2$ or $2\lambda^2$; and similarly those of $\nu^2\lambda^2$, $\lambda^2\mu^2$ are equal to $2\mu^2$, $2\nu^2$ respectively.

2. Since $\mathrm{I}=\frac{1}{24}a^2\left\{(m-n)^2+(n-l)^2+(l-m)^2\right\}$ or $\frac{1}{24}a^2(\lambda^2+\mu^2+\nu^2)$,

$$3J = \tfrac{1}{144}a^3(2l-m-n)(\ldots)(\ldots) = \tfrac{1}{144}a^3(\mu-\nu)(\nu-\lambda)(\lambda-\mu)$$

$$= -\tfrac{1}{144}a^3\left\{(\mu-\nu)\lambda^2+(\nu-\lambda)\mu^2+(\lambda-\mu)\nu^2\right\} = \tfrac{1}{144}a^3\left\{(2l-m-n)\lambda^2+\ldots\right\},$$

and $c = \frac{1}{6}a(l+m+n)$; therefore

$$J' = 3J + cI = \tfrac{1}{144}a^3\left\{(2m+2n-l)\lambda^2+(2n+2l-m)\mu^2+(2l+2m-n)\nu^2\right\}.$$

Again, $\quad IK = I\left\{6cJ' + I(I-3c^2)\right\}$

$$= I\left\{6cJ'+I(ae-4bd)\right\} = I\left\{a(l+m+n)J'-\tfrac{1}{4}a^2I(mn+nl+lm)\right\}$$

$$= \tfrac{1}{8912}a^6(\lambda^2+\mu^2+\nu^2)\Big[2(l+m+n)\left\{(2m+2n-l)\lambda^2+\ldots\right\} \\ -3(mn+nl+lm)(\lambda^2+\mu^2+\nu^2)\Big].$$

Hence $\quad \Delta = IK - 3J'^2$

$$= \tfrac{1}{8912}a^6\left\{\Sigma(2m+2n-l)\lambda^2.\,\Sigma(2m+2n+2l-2m-2n+l)\lambda^2-3\Sigma mn.\,\Sigma\lambda^2\right\}$$

$$= \tfrac{1}{2304}a^6\left\{\Sigma(2m+2n-l)\lambda^2.\,\Sigma l\lambda^2-\Sigma mn.\,\Sigma\lambda^2\right\}.$$

But in terms of the roots $\Delta = \frac{1}{256}a^6\lambda^2\mu^2\nu^2$; hence $\Delta = IK-3J'^2$ is tantamount to the identity 2.

3378. (Proposed by the Rev. T. P. Kirkman, M.A., F.R.S.)—

What a sight was there
In that summer air,
With the pomp of stately trees,
And the hum of the plundering bees,
And the fragrance of the flowers
By the decorated bowers
On Mr. Punch's lawn,
With jewelled ladies passing fair,
And Field-Marshals,
And Admirals,
And spectacled Professors,—
No painter ever yet has drawn,
And you will never guess, Sirs.

Mr. Punch comes out
And walks about
With Judy, who, with air didactic,
Carries a little bag.
Quoth she, "The croquet seems to flag;
I'll give you a lesson in Tactic.

With every left hand in a right,
Form circles of faces, as you please,
Of A's or B's, or twos or threes,
And hold the captives tight;
And whoso chooses alone to stand
Will clasp his left in his own right hand:
A circle of one
Is that simpleton.

Your number is R; you have formed a partition
Of R into a A's and b B's, and so on,
In numbers descending A, B, C ...,
Where A the biggest at least is three:
So that's your position,
And now we can go on.

From this bag, which holds you all,
At random I draw two names, and call
M, N: then mark, whoever clasps
The left of either lets go it, and grasps
The left of the other: the trumpets peal,
And every circle dances round,
While the broken ones gracefully change their ground,
And the solitaires twirl on toe or heel.

You are circles again: from the lessened store
At random I draw a couple more.

'Tis a curious skill
To predict at will
The circles and couples the word to fulfil.

Of random couples thus be cried
A random number k;
k times with music modified
The whirling circles play:

In terms of the given aA, bB ...
Say, what's the chance
In the final dance
The circles be all of under three—
All couples or capering unity?"

3402. (Proposed by the Rev. T. P. Kirkman, M.A., F.R.S.)—Let $N = aA + bB + cC + \ldots$ $(A>2)$ be any partition of N, and let Θ be any substitution having a circular factors of A, b of B, c of C elements, &c.;

then the number of ways in which Θ can be broken into the product $\theta\theta' = \Theta$ of two substitutions of the second order is

$$\Sigma_{\alpha\beta\gamma\ldots} \frac{A^{a-\alpha} B^{b-\beta} C^{c-\gamma} \ldots\ldots \Pi a \, \Pi b \, \Pi c \ldots\ldots}{2^{\alpha+\beta+\gamma\ldots} \Pi(a-2\alpha)\, \Pi(b-2\beta)\, \Pi(c-2\gamma) \ldots\ldots \Pi\alpha \, \Pi\beta \, \Pi\gamma \ldots\ldots},$$

where $\alpha, \beta, \gamma \ldots$ are any numbers, zero or positive.

Solution by the PROPOSER.

Let Θ be any substitution of the order k, made with N elements on the partition $\qquad N = aA + bB + Cc + \ldots\ldots jJ,$

where $A > 2$ and $J \overline{\gtrless} 1$, having a circular factors of A elements, b of B elements, &c.; and let ϕ be any substitution of the second order, that is, made by one or more transpositions of two elements; then we have to find the chance that $\phi\Theta = \phi'$ shall be a substitution ϕ' of the second order, or the chance that $\Theta = \phi\phi'$ is a product of two of the second order. The entire number Q of possible substitutions ϕ of that order is given, and easily assigned; the number of ways in which Θ can be broken into a product $\phi\phi'$ is given in Question 3402; calling this P, the answer to Judy's Question is P : Q.

I have shown, in Arts. 25, 26 of my *Memoir on the Theory of Groups*, Theorem G, that there are $V = \Pi N : (R_k \,.\, k \,.\, \Pi a \, \Pi b \ldots \Pi j)$ groups made in the above partition of N, each consisting of k powers of a substitution Θ and k substitutions of the second order $q_1 q_2 \ldots q_k$, such that $\Theta = q_1 q_2 = q_2 q_3 = \ldots q_k q_1$, $\Theta^2 = q_1 q_3 = q_2 q_4 = \ldots q_k q_2$, $\Theta^3 = q_1 q_4 = q_2 q_5 = \ldots q_k q_3$, &c., and $\Theta^{k-1} = q_1 q_k = q_2 q_1 =$ &c. The number of different groups of powers of substitutions (Θ) of the order k on this partition is (Art. 19, *ubi supra*) $W = \Pi N : (R_k \, \Pi a \, \Pi b \ldots \Pi j \, A^a B^b \ldots J^j)$, so that our Θ will be found in $V : W = k^{-1} . A^a B^b \ldots J^j$ of the groups of Theorem G, having written under its k powers this number of different sets of didymous radicals, as I call them; that is, there are $k^{-1} A^a B^b \ldots J^j$ different sets $q_1 q_2 \ldots q_k$, giving for our Θ each k decompositions of the form $\phi\phi_1$, viz.,

$$\Theta = q_1 q_2 = q_2 q_3 = q_3 q_4 = \ldots = q_k q_1.$$

We have thus $A^a B^b \ldots J^j$ of these decompositions. But these are all of the form described at page 62 of Vol. XV. of the *Reprint*, Quest. 3184. The method given at page 63 augments the decompositions of Θ to the full number

$$P = \Sigma_{\alpha\beta\gamma\ldots} \frac{A^{a-\alpha} B^{b-\beta} C^{c-\gamma} \ldots \Pi a \, \Pi b \, \Pi c \ldots}{2^{\alpha+\beta+\gamma+\ldots} \Pi(a-2\alpha)\, \Pi(b-2\beta)\, \Pi(c-2\gamma) \ldots \Pi\alpha \, \Pi\beta \, \Pi\gamma \ldots},$$

where $\alpha, \beta, \gamma, \ldots$ are any and all numbers positive or zero. For taking any 2α of the a factors of the order A, transposing pairs of them, and effecting the vertical cyclical permutations of subindices indicated in page 63 of Vol. XV. of the *Reprint* [where, in line 14, we should read c_3 for c_2], we merely substitute for A^a in the product $A^a B^b C^c \ldots$ the factor

$$\frac{A^{a-2\alpha} . \Pi a}{\Pi(a-2\alpha)\, \Pi 2\alpha} \left\{1 . 3 . 5 \ldots (2\alpha - 1)\right\} A^{\alpha}.$$

By writing this solution P of Question 3402 over the given number Q, we have the answer to Judy's problem.

3334. (Proposed by the Rev. R. TOWNSEND, M.A., F.R.S.)—Prove that the volumes of any tetrahedron, and of the inscribed ellipsoid which touches at the centres of gravity of its four faces, have the same principal axes at their common centre of gravity; and that their moments of inertia for all planes passing through that point have the same constant ratio (viz., $18\sqrt{3} : \pi$) to each other.

Solution by the PROPOSER.

For, the twelve systems of three planes passing through their common centre of gravity O, one parallel to any face of the tetrahedron, and the other two passing through the opposite vertex, one parallel to any side and the other passing through the opposite vertex of the face, forming evidently for each volume a common triad of planes for which $\Sigma(yzdm)=0$, $\Sigma(zxdm)=0$, $\Sigma(xydm)=0$, therefore, &c. as regards the coincidence of the principal axes and the constancy of the ratio of the several moments of inertia for both.

To determine the value of the constant ratio, since, for the plane through O parallel to the face A, $\Sigma(ma^2)$ for the tetrahedron $= \frac{3}{5}Vp^2$, and for the ellipsoid $= \frac{1}{5}V'p^2$, where p is the perpendicular from O on A; and since $V : V' = 6\sqrt{3} : \pi$, therefore $I : I' = 18\sqrt{3} : \pi$, and therefore, &c.

3010. (Proposed by J. J. WALKER, M.A.)—The equation to any quadric surface of revolution, referred to three rectangular axes, may be thrown into the form

$$(a^2-k^2)\,x^2+(b^2-k^2)\,y^2+(c^2-k^2)\,z^2+2bcyz+2acxz+2abxy$$
$$+2dx+2ey+2fz+g=0.$$

Solution by the PROPOSER.

For these values of the coefficients of the homogeneous part of the equation satisfy identically the condition that the discriminating cubic should have equal roots and contain four independent constants a, b, c, k. But the coefficients in the homogeneous part of the equation

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2C'yz+\ \dots\dots=0$$

contain only four independent constants, since they also must satisfy the same condition. Hence, if a, b, c, k be determined by the equations $(A=1)$ $b^2-k^2=A'(a^2-k^2)$, $c^2-k^2=A''(a^2-k^2)$, $bc=B(a^2-k^2)$, $ac=B'(a^2-k^2)$, we must have $ab\equiv B''(a^2-k^2)$.

3483. (Proposed by Professor WOLSTENHOLME.)—A point is determined in space by taking at random its distances from three given points A, B, C; prove that the density at any point varies as

$$(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}},$$

α, β, γ being the angles subtended at the point by the sides of the triangle ABC.

Solution by G. S. Carr.

The condition will be fulfilled by supposing the distances from A, B, C to vary by successive increments δx, δy, δz, having always the same absolute ultimate values.

The density will then vary as the volume of the parallelepiped on δx, δy, δz which exists for each point, and has α, β, γ for the angles between its edges. But this volume is known to be

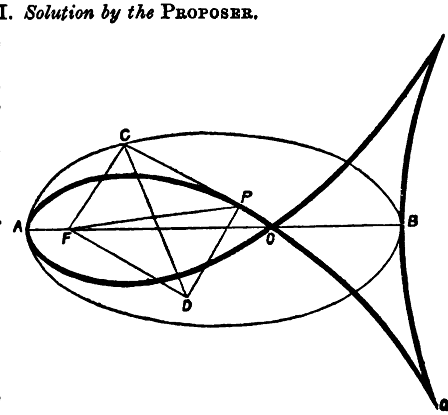
$$\delta x \,.\, \delta y \,.\, \delta z\,(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}},$$

which proves the proposition.

[The analogous theorem for a plane is Prof. Crofton's Quest. 2398, of which two solutions are given on pp. 33—34 of Vol. XII. of the *Reprint*.]

3430. (Proposed by the Editor.)—Find the equation, form, length, and area of the first negative focal pedal of an ellipse or a parabola (that is, the envelope of perpendiculars at the ends of the focal radii.

I. *Solution by the* Proposer.



1. Let AFB be the major axis of the ellipse, FC a focal radius, CP a perpendicular to FC at its extremity C, and P the point on the pedal corresponding to C on the ellipse; then the envelope of CP is the first negative pedal of the ellipse.* Draw the normal CD to the ellipse at C, and FD, the polar sub-normal, perpendicular to FC. Then D is the instantaneous centre, about which, for the moment, any point in the revolving angle PCF may be supposed to turn; for CD, FD are normals to the paths of two points (C, F) in this angle, of which C moves in the curve of the ellipse, and F, for the instant, along the line CF. Hence the normal to CP and its envelope, at the point of contact P, must also pass through D; therefore P is the foot of the perpendicular from D on CP, and the figure CFDP is a rectangle. This gives us a simple *construction* for the point on the pedal corresponding to any point on the ellipse.

We may hence define the pedal as a locus, and in a very simple way by enunciating it in a mechanical form, as follows,—

Two forces, which act at a focus (F) of an ellipse, are represented, both in magnitude and direction, by the focal radius (FC) and polar subnormal (FD) of a point

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* Articles on pedals, by the Editor, will be found on pp. 23, 35 of Vol. I. of the *Reprint*.

(C) in the curve; the locus of the extremity (P) of their resultant is the first negative pedal of the ellipse.

2. Let a, b, h, e be the principal semi-axes, semi-parameter, and eccentricity of the ellipse; and put $FP = r$, $FC = p$, $PC = u$, $\angle PFB = \theta$, $\angle CFB = \phi$, $\psi =$ the eccentric angle of the point C on the ellipse, and (x, y) the rectangular coordinates of the corresponding point P on the pedal, referred to F as origin, and FB as positive axis of x.

3. Then we readily obtain the following system of equations:—

$$p = \frac{h}{1-e\cos\phi} = a(1+e\cos\psi) = \frac{b\sin\psi}{\sin\phi} = \frac{a(e+\cos\psi)}{\cos\phi} \quad\dots\dots\dots (1),$$

$$u = \frac{ep^2\sin\phi}{h} = \frac{ebp\sin\psi}{h} \quad\dots\dots\dots (2),$$

$$x = p\cos\phi + u\sin\phi = \frac{h(\cos\phi - e\cos 2\phi)}{(1-e\cos\phi)^2} = a\cos\psi(1-e\cos\psi) + 2ae \dots (3),$$

$$y = p\sin\phi - u\cos\phi = \frac{h(\sin\phi - e\sin 2\phi)}{(1-e\cos\phi)^2} = \frac{a^2}{b}\sin\psi(1-2e\cos\psi+e^2) \dots (4),$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\phi(1-2e\cos\phi)}{\cos\phi - e\cos 2\phi} \quad\dots\dots\dots (5),$$

$$r^2 = \frac{p^4}{h^2}(1-2e\cos\phi+e^2) = \frac{p^3}{h}(1-e\cos\psi) = \frac{(2a-p)p^3}{b^2} \quad\dots\dots (6),$$

$$\frac{d\theta}{d\phi} = \frac{1-3e\cos\phi+2e^2}{1-2e\cos\phi+e^2},\quad \frac{d\theta}{d\psi} = \frac{b(1-2e\cos\psi)}{p(1-e\cos\psi)},\quad \frac{d\phi}{d\psi} = \frac{b}{p} \quad\dots\dots (7),$$

$$\frac{dp}{dr} = \frac{b^2 r}{(3a-2p)p^2} \quad\dots\dots\dots (8).$$

4. These equations are sufficient for a very full discussion of the properties of the pedal. The first part of equation (1), viz., $p(1-e\cos\phi) = h$, may be appropriately called the tangential polar equation, or pedal equation, of the envelope of CP; and from (6) we obtain $b^2r^2 = (2a-p)p^3$, which is the equation of the pedal in terms of the radius-vector (r) and the perpendicular (p) from the pole on the tangent.* In equations (3) and (4) the rectangular (x, y) coordinates of the point P on the pedal corresponding to the point C on the ellipse are given as functions of ϕ or of ψ, and (5) and (6) express similarly the polar (r, θ) coordinates of P. The length and area may be expressed in terms of ϕ or ψ by equations (3) to (7), and the direction of curvature is given by (8).

5. The equation in x and y of the pedal may be readily found by eliminating ψ (or ϕ) from (3) and (4). Putting ϵ^2 for $1-e^2$, the equation may be put into the following form:—

$$\epsilon^2(x^2+\epsilon^2y^2)^2 - 2(5-4e^2)ehx^3 - 2(5+4e^2)\epsilon^2ehxy^2 - (1-32e^2+16e^4)h^2x^2$$
$$-(1-20e^2-8e^4)h^2y^2 + 8(1-4e^2)eh^3x - 16e^2h^4 = 0 \quad\dots\dots\dots (9).$$

When $e=1$, the parameter $2h$ remaining constant, the ellipse becomes a parabola, and the equation (9) of the pedal takes the simple forms

$$2(x+2h)^3 = 27(x^2+y^2),\ \text{or}\ 2r\cos^3\tfrac{1}{3}\theta = h \quad\dots\dots\dots (10),$$

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* In Art. 5 of the EDITOR's paper on pedals (*Reprint*, Vol. I., p. 24), it is shown that if $f(r, p) = 0$ be the equation of the primitive curve, that of its first negative pedal will be $f\left(p, \frac{p^2}{r}\right) = 0$. Thus, the focal equation in (r, p) of the ellipse being $b^2r = (2a-r)p^2$, that of the first negative pedal is $b^2r^2 = (2a-p)p^3$, which is identical with the equation otherwise deduced above.

the second of these forms being obtained by transforming the first to polar coordinates, by putting $x = -r\cos\theta$ and $y = r\sin\theta$, the positive direction of θ being estimated from the line FA.*

6. In the case of the parabola, *the first negative pedal is identical with the catacaustic for rays perpendicular to the axis.* For, by a well-known property of the parabola, any such incident ray makes, with the tangent at the point of incidence, an angle equal to that between the normal and the focal radius; hence the reflected ray, which has the same inclination to the normal as the incident ray, must make, with the focal radius, an angle equal to that between the tangent and normal, that is, a right-angle. Thus the catacaustic is the envelope of perpendiculars at the extremities of focal radii; that is to say, it is the first negative pedal of the parabola, and has for its equation either of the forms (9) given in the preceding article.

7. In Dr. BOOTH's system of tangential coordinates, the equation of a curve is of the form $\qquad U \equiv f(\xi, v) = 0,$
where ξ and v are the reciprocals of the intercepts made by a tangent on the axes of x and y respectively. Hence, for the same origin, the tangential coordinates (ξ, v) and the projective (or Cartesian) coordinates (x, y) of the point of contact of the tangent are connected by the relations

$$x\xi + yv = 1 \text{ and } \frac{x}{y} = \frac{\xi}{v}, \quad \text{or } x = \frac{\xi}{\xi^2+v^2} \text{ and } y = \frac{v}{\xi^2+v^2}.$$

Substituting these values in the projective equation of the ellipse for origin F, viz., $\qquad b^2(x-ae)^2 + a^2y^2 = a^2b^2,$
the Boothian tangential equation for this origin is found to be

$$U \equiv \xi^2 + v^2 - \left\{e\xi + h(\xi^2+v^2)\right\}^2 = 0 \quad\text{.....................}(11).$$

Since $\xi = \dfrac{\cos\phi}{p}$, and $v = \dfrac{\sin\phi}{p}$, (11) gives us at once

$$(h + ep\cos\phi)^2 = p^2, \quad \text{or } p = \frac{h}{1-e\cos\phi},$$

which is identical with our pedal equation (1). Of course, U may have been otherwise formed from (1) by a reversal of this last process of substitution.

Now it has been shown by Dr. BOOTH, in his solution of Question 1509,† that the projective equation can be formed from the tangential equation U, by eliminating ξ and v from the equations

$$U=0, \quad \frac{dU}{x\,d\xi} = \frac{dU}{y\,dv} = \frac{\xi\,dU}{d\xi} + \frac{v\,dU}{dv};$$

hence we have thus another method of obtaining the equation in x and y of the pedal.‡

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* The same equation (9) for the pedal of the parabola may be otherwise obtained from (6) and (7). For when $e=1$, (7) gives $\dfrac{d\phi}{d\theta} = \dfrac{2}{3}$; hence, estimating these angles from FA, so that θ and ϕ may vanish simultaneously, we have $\phi = \frac{2}{3}\theta$. Substituting this value of ϕ in (6), and changing the sign of $\cos\phi$, we have

$$r^3(1+\cos\tfrac{2}{3}\theta)^3 = 2h^3, \quad \text{or } 2r\cos^3\tfrac{1}{3}\theta = h.$$

† See *Reprint*, Vol. II., pp. 20—22, and also pp. 1—3. In an Editorial note to the solution of Question 1509, the projective equation of the catacaustic of the circle for parallel rays is found by this process of elimination.

‡ This elimination we have proposed as Question 3550.

8. We proceed next to trace the pedal, and find its length and area.

When $e > \frac{1}{2}$, put $\frac{1}{2e} = \cos\alpha$ and $\frac{1}{1-2e^2} = \cos\beta$; thus

$$\beta > \alpha, \text{ and } \cos\alpha \cos\beta = \cos 2\alpha.$$

Then, while ψ is between the limits $(0, \alpha)$, y and θ are negative and decreasing, but x and r are increasing; also, $2p$ being greater than $3a$, $\frac{dp}{dr}$ is negative, and therefore the curve is convex towards the pole F. In this interval, the part generated is BQ. At Q, when $\psi = \alpha$ or $p = \frac{3}{2}a$, x and r are each a maximum, y and θ each a minimum, $\frac{dp}{dr}$ vanishes, and a ceratoid cusp is formed. From $\psi = \alpha$ to $\psi = \pi$, $\frac{dp}{dr}$ is positive, and therefore the curve is concave towards F; moreover, it crosses the axis AB at the point O, where $\psi = \beta$, $p = 2h$, $x = r = \text{FO} = 4eh$, and the mixtilineal angle $\text{BOQ} = \frac{1}{2}\pi - \alpha$.

Hence the pedal is of the form shown in the figure. It is symmetrical on both sides of the axis AFOB, and consists of a loop on AFO as diameter, and a curvilinear triangle OQR having a double point at O, and two ceratoid cusps at Q and R.

9. When e does not exceed $\frac{1}{2}$, the curvilinear triangle OQR disappears, and the pedal consists of the loop alone, which then has AB for diameter.

When $e = 0$, both the ellipse and its pedal become the circle on BC as diameter.

When $e = 1$, the parameter $2h$ remaining constant, the ellipse becomes a parabola, and OQ, OR become infinite branches.

10. Let Ω be the rectangular, and Σ the sectorial area of the pedal; and for shortness' sake put

$$\mu = \frac{2-3e^2}{2(1-e^2)^{\frac{1}{2}}} = \frac{(2-3e^2)\,a}{2b} = \frac{3h-a}{2b}, \quad \nu = \frac{(16e^4-26e^2+13)(4e^2-1)^{\frac{1}{2}}}{6};$$

then we shall have

$$\Omega = \int y\,dx = \frac{a^3}{b}\int\left\{2e^2\sin^2\psi + e(3-4e^2)\cos\psi - 1\right\}\sin^2\psi\,d\psi$$

$$= \tfrac{1}{4}\mu a^2(\sin 2\psi - 2\psi) + \frac{ea^3}{6b}(6-8e^2-3e\cos\psi)\sin^3\psi \;\;\ldots\ldots\ldots\ldots (12),$$

$$\Sigma = \int \tfrac{1}{2} r^2 d\theta = \frac{a^3}{2b}\int(1-3e^2\cos^2\psi - 2e^3\cos^3\psi)\,d\psi$$

$$= \tfrac{1}{2}\mu\psi a^2 - \frac{e^2a^3}{12b}(8e + 9\cos\psi + 4e\cos^2\psi)\sin\psi \ldots\ldots\ldots\ldots\ldots (13).$$

11. When e does not exceed $\frac{1}{2}$, the area of the whole pedal, which is then the loop on BC, will be obtained by taking the double of Ω between the limits $\psi = \pi$ and $\psi = 0$, which correspond to the *increase* of x, or of Σ between the limits 0 and π of ψ; and they both give $\mu\pi a^2$ for the entire area.

In other cases (when $e > \frac{1}{2}$) $\mu\pi a^2$ is the excess of the positive over the negative area, that is to say, of the loop APO over the triangle OQR. For in the integration of Ω the elements summed are positive from $\psi = \pi$ to $\psi = \beta$; negative from $\psi = \beta$ to $\psi = \alpha$, since y is then negative and dx positive; and positive again from $\psi = \alpha$ to $\psi = 0$, since y and dx are then

both negative. Also, in the integration of Σ, the elements summed are negative from $\psi=0$ to $\psi=\alpha$, positive from $\psi=\alpha$ to $\psi=\beta$,—giving thus far the negative difference as the area of OQR,—and positive still from $\psi=\alpha$ to $\psi=\pi$.

12. When $e>\frac{1}{2}$, we may find the positive area of the triangle OQR by taking Ω between the 0 and β of ψ, or Σ between the limits β and 0. We thus obtain

$$\text{area of triangle OQR} = (\nu-\mu\beta)\,a^2.$$

The area of the loop OPA, which is found by adding $\mu\pi a^2$ to the area of OQR, will be greater or less than this according as μ is positive or negative, that is to say, according as $e^2 >$ or $< \frac{2}{3}$.

When $e^2=\frac{2}{3}$, or $a^2=3b^2$, or $a=3h$, we have $\mu=0$, and then

$$\text{area of OQR} = \text{area of OAP} = \nu a^2 = (\tfrac{25}{18}\sqrt{15})\,h^2.$$

13. Let s be the length of an arc of the pedal; then $pd\phi$ is the increment of $s+u$ between two consecutive values of p; hence, by (8), we have

$$s+u=\int p\,d\phi=\int b\,d\psi=b\psi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14).$$

When $e<\frac{1}{2}$, the length of the whole curve, which then consists of the loop on AB alone, is found by taking the double of the integral (14) between the limits 0 and π of ψ; and since, in this case, u vanishes at both limits, the whole length of the pedal is $2\pi b$, or equal to that of the circle on the minor axis of the ellipse as diameter.

When $e>\frac{1}{2}$, the values (u_1, u_2 say) of u at the points Q (when $\psi=\alpha$) and O (when $\psi=\beta$) are, respectively,

$$u_1=\frac{3a^2}{4b}(4e^2-1)^{\frac{1}{2}}=\frac{3a^2}{4b}\tan\alpha,\quad u_2=2h\,(4e^2-1)^{\frac{1}{2}}=2h\tan\alpha;$$

and thus the separate lengths of the several parts of the curve are found to be

$$\text{QOPAOR (part concave towards F)}=\frac{3a^2}{2b}\tan\alpha+2b\,(\pi-\alpha),$$

$$\text{QBR (part convex towards F)}=\frac{3a^2}{2b}\tan\alpha-2b\alpha,$$

$$\text{loop OPAO}=4h\tan\alpha+2b\,(\pi-\beta)=4\,(4e^2-1)^{\frac{1}{2}}\,h+2b\cos^{-1}\left(\frac{2e^2-1}{e}\right).$$

14. When $e=1$, or the ellipse becomes a parabola, the expressions in Arts. 11 and 13 for the area and the length of the loop take indeterminate forms, but their limiting values give*

$$\text{length of loop of parabolic pedal} = (6\sqrt{3})\,h,$$
$$\text{area of loop of parabolic pedal} = (\tfrac{12}{5}\sqrt{3})\,h^2.$$

15. When the directing curve is a parabola, it has been shown in Art. 6 that the polar equation of the pedal is $2r\cos^3\frac{1}{3}\theta=h$. Hence, in this case, the expressions for the length and area of the loop given in Art. 14, may be otherwise obtained as follows:—

$$\text{Length}=2\int_0^{\pi}\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}d\theta=h\int_0^{\pi}\sec^4\tfrac{1}{3}\theta\,d\theta$$

$$=h\left(\tan^3\tfrac{1}{3}\theta+3\tan\tfrac{1}{3}\theta\right)_{\theta=0}^{\theta=\pi}=(6\sqrt{3})\,h;$$

* These limiting values have been deduced from the general expressions, by the EDITOR, in his Solution of Question 1331, *Reprint*, Vol. I., p. 26.

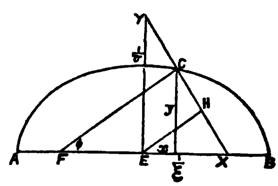
$$\text{Area} = \int_0^\pi r^2 d\theta = \tfrac{1}{4}h^2 \int^\pi \tfrac{1}{3}\theta \sec^6 d\theta$$

$$= \tfrac{1}{4}h^2 \left(3\tan\tfrac{1}{3}\theta + 2\tan^3\tfrac{1}{3}\theta + \tfrac{3}{5}\tan^5\tfrac{1}{3}\theta\right)_{\theta=0}^{\theta=\pi} = (\tfrac{16}{5}\sqrt{3})\, h^2.$$

16. The polar equation of the locus of the instantaneous centre (G), or, what is the same thing, that of the end (G) of the focal polar subnormal of the ellipse, is at once seen from (1) and (2) to be

$$r = \frac{eh\cos\theta}{(1-e\sin\theta)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

II. *Solution by the* Rev. Dr. Booth, F.R.S.



1. Let the equation of the ellipse be

$$b^2x^2 + a^2y^2 = a^2b^2 \ldots\ldots\ldots\ldots (1),$$

and the dual equation $x\xi + yv = 1 \ldots\ldots (2)$;

then $\tan\phi = \dfrac{y}{ae+x} = \cot X = \dfrac{v}{\xi}$,

therefore $\xi y = aev + vx \ldots\ldots\ldots\ldots (3)$.

Combining this with (2), we have

$$x = \frac{\xi - aev^2}{\xi^2+v^2},\quad y = \frac{v + ae\xi v}{\xi^2+v^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Substituting these values of x and y in (1), we get the tangential equation of the pedal when the origin is at the centre E of the ellipse, viz.

$$a^2v^2 + b^2\xi^2 + 2a^3e^3\xi v^2 + a^2e^2v^2(a^2\xi^2+b^2v^2) = a^2b^2(\xi^2+v^2)^2 \ldots\ldots\ldots (5).$$

When $v=0$, or the tangent is vertical, $\dfrac{1}{\xi} = \pm a$, and when $\xi = 0$, or the tangent is horizontal, $\dfrac{1}{v} = \pm\dfrac{b^2}{a} = \pm h$, as they should be.

The equation is satisfied by $v=0$, $\xi=0$; hence the curve has branches touched at infinity by the axes of coordinates.

The rectification of this curve is easily effected. Multiply both sides of the equation by p^4, p being the perpendicular EH from the origin on the tangent CX to the pedal; then, since $p\xi = \cos\phi$, $pv = \sin\phi$, we have

$$(1-e^2\cos^2\phi)p^2 + 2(ae^3\sin^2\phi\cos\phi)p = b^2 - a^2e^2\sin^2\phi(1-e^2\sin^2\phi) \ldots (6),$$

which is the tangential polar equation of the pedal, the centre E of the ellipse being the origin. Hence p is found by a quadratic equation. Solving it, we have

$$p = \frac{-ae^3\cos\phi\sin^2\phi \pm a(1-e^2)}{1-e^2\cos^2\phi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Hence the expression for rectification is easily integrated, namely, $\int p\,d\phi$.

Let p_1 and p_2 be the roots of the equation (6), then we have

$$p_1 - p_2 = \frac{2b^2}{a(1-e^2\cos^2\phi)} = \text{focal chord CFC}' \ldots\ldots\ldots\ldots (8).$$

2. Next, for the parabola, let the focus F be taken as the origin of projective coordinates, the axis of the parabola being the axis of x. The

equation of the parabola is $y^2 = 4a(a+x)$; and (4) here becomes

$$x = \frac{\xi}{\xi^2+v^2},\ y = \frac{v}{\xi^2+v^2}.$$

hence, by substitution we obtain

$$4a^2(\xi^2+v^2)^2+4a\xi(\xi^2+v^2)-v^2 = 0 \ \ldots\ldots\ldots\ldots\ \ldots\ldots (9),$$

which is the tangential equation of the pedal of the parabola, the focus F being the origin.

The rectification of this curve presents no difficulty. Multiplying by p^4 (p being here FC), the resulting equation becomes

$$4a^2+4a\cos\phi\,.\,p-p^2\sin^2\phi = 0,\ \text{ whence } p = \frac{2a}{\pm 1-\cos\phi}.$$

Hence $pd\phi$ is easily integrable.

3431. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Find the equation, form, length, and area of the first negative central pedal of an ellipse (that is, the envelope of perpendiculars at the ends of central radii).

I. *Solution by* STEPHEN WATSON.

If ψ be the eccentric angle at any point on the ellipse, the equation of a perpendicular at that point to the radius vector drawn to the point is

$$by\sin\psi+ax\cos\psi = c^2\cos^2\psi+b^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Differentiating with respect to ψ, we have

$$by\cos\psi-ax\sin\psi = -2c^2\sin\psi\cos\psi \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$(1)^2+(2)^2$ gives $3c^4\cos^4\psi-2c^2m^2\cos^2\psi+X^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$

where $m^2 = a^2+c^2$, and $X^2 = a^2x^2+b^2y^2-b^4$.

Also (1) $\cos\psi-(2)\sin\psi$ gives $c^2\cos^3\psi-m^2\cos\psi = -ax \ \ldots\ldots\ldots (4);$

hence $3(4)^2-(3)\cos^2\psi$ gives

$$4c^2m^2\cos^4\psi+(X^2-3m^4)\cos^2\psi+3a^2x^2 = 0 \ldots\ldots\ldots\ldots\ldots (5).$$

Eliminating $\cos^2\psi$ from (3) and (5), we have the equation of the pedal, viz.,

$$(4m^2X^2-9a^2c^2x^2)^2+(3X^2-m^4)(X^4-3m^4X^2+6a^2c^2m^2x^2) = 0 \ldots (6).$$

For the purpose of tracing the curve, finding its area, &c., equation (4) and $\quad c^2\sin^3\psi+(b^2-c^2)\sin\psi = by \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$

which may be similarly obtained from (1) and (2), are most convenient. Thus, one-fourth of the area must lie between $\psi=0$, $\psi=\frac{1}{2}\pi$; therefore

$$\text{Area} = 4\int_0^a ydx = \frac{4}{ab}\int_{\frac{1}{2}\pi}^{0}\left\{c^2\sin^3\psi+(b^2-c^2)\sin\psi\right\}$$

$$\times(3c^2\cos^2\psi-m^2)\sin\psi\,d\psi = \frac{\pi}{8ab}(10a^2b^2-a^4-b^4).$$

II. *Solution by the* EDITOR.

1. Adopting the method and notation of the second solution of the foregoing Question (3430), the point on the *central* pedal corresponding to any point on the ellipse may be determined by a similar construction to that which is there given for the *focal* pedal; moreover, in this case, the equation-system will be as follows:—

$$p = \frac{b}{(1-e^2\cos^2\phi)^{\frac{1}{2}}} = a\,(1-e^2\sin^2\psi)^{\frac{1}{2}} = \frac{a\cos\psi}{\cos\phi} = \frac{b\sin\psi}{\sin\phi} \quad\text{.........}\ (1),$$

$$u = \frac{e^2p^3}{b^2}\sin\phi\cos\phi = \frac{ae^2p}{b}\sin\psi\cos\psi \quad\text{..................}\ (2),$$

$$r^2 = p^2+u^2 = \frac{b^2\,(1-e^2\cos^2\phi+e^4\sin^2\phi\cos^2\phi)}{(1-e^2\cos^2\phi)^3}$$

$$= p^2\left(1+\frac{a^2e^4}{b^2}\sin^2\psi\cos^2\psi\right) \quad\text{..............................}\ (3),$$

$$x = p\cos\phi+u\sin\phi = \frac{b\cos\phi\,(1-e^2\cos 2\phi)}{(1-e^2\cos^2\phi)^{\frac{3}{2}}}$$

$$= a\cos\psi\,(1+e^2\sin^2\psi) \quad\text{.......................................}\ (4),$$

$$y = p\sin\phi-u\cos\phi = \frac{b\sin\phi\,(1-2e^2\cos^2\phi)}{(1-e^2\cos^2\phi)^{\frac{3}{2}}}$$

$$= b\sin\psi\left(1-\frac{a^2e^2}{b^2}\cos^2\psi\right) \quad\text{.........}\ (5),$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\phi\,(1-2e^2\cos^2\phi)}{\cos\phi\,(1-e^2\cos 2\phi)} = \frac{b\sin\psi\,(1-a^2b^{-2}e^2\cos^2\psi)}{a\cos\psi\,(1+e^2\sin^2\psi)} \quad\text{......}\ (6),$$

$$\frac{d\theta}{d\psi} = \frac{a^3}{br^2}\left\{e^4\sin^2\psi\,(1-3\cos^2\psi)+2e^2\cos^2\psi-1\right\} \quad\text{............}\ (7).$$

2. From the preceding equations the properties of the pedal may be readily developed. Equations (4) and (5) give x and y as functions of ψ, and correspond with Mr. WATSON's equations (4) and (7). By eliminating ψ from these equations, as given in the foregoing solution, we obtain the equation in x and y of the pedal. The form of the pedal is easily determined. The curve lies wholly inside the ellipse, which it touches at the ends of both axes, and is at all points concave towards the centre.

3. From (1) and (3) we obtain

$$a^2b^2r^2 = (a^2+b^2-p^2)\,p^4 \quad\text{,..........................}\ (8),$$

which is the equation of the pedal in terms of the radius vector (r) of any point on it, and the perpendicular (p) from the origin on the tangent at that point. This equation may be otherwise obtained by the aid of the general property referred to in the note to Art. 4, p. 78, of the EDITOR's solution of Quest. 3430. For the central equation in r and p of the ellipse is $(a^2+b^2-r^2)\,p^2 = a^2b^2$; hence, putting in this p for r, and $\frac{p^2}{r}$ for p, we obtain the equation (8) of the central pedal.

4. The area of the pedal may be obtained from (7), thus:—

$$\text{Area} = 2\int_0^{\pi}\tfrac{1}{2}r^2d\theta = \frac{a^3}{8b}\int_0^{\pi}d\psi\left\{8\,(1-e^2)-e^4-4e^2\,(2-e^2)\cos 2\psi - 3e^4\cos 4\psi\right\}$$

$$= \pi\left(ab-\frac{a^3e^4}{8b}\right),$$

which, in a slightly different form, agrees with Mr. WATSON's result.

Hence the area between the pedal and the ellipse is $\frac{\pi a^3 e^4}{8b}$.

5. Let (α, β) be the point on the ellipse corresponding to (x, y) on the pedal; then $\alpha = a\cos\psi$ and $\beta = b\sin\psi$; hence (4) and (5) become

$$\frac{x}{\alpha} = 1 + \frac{e^2\beta^2}{b^2},\quad \frac{y}{\beta} = 1 - \frac{e^2\alpha^2}{b^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

and from (9) we obtain the following equations:—

$$\frac{a^2x}{\alpha} - \frac{b^2y}{\beta} = 2a^2e^2,\quad \frac{x}{\alpha} - \frac{y}{\beta} = \frac{e^2}{b^2}(\alpha^2+\beta^2)\ldots\ldots\ldots\ldots\ldots\ldots (10);$$

$$\frac{x}{\alpha} + \frac{e^2\alpha^2}{a^2} = 1+e^2,\quad \frac{y}{\beta} - \frac{e^2\beta^2}{h^2} = \frac{1-2e^2}{1-e^2} \quad\ldots\ldots\ldots\ldots\ldots (11).$$

By eliminating α and β from either of the pairs (9), (10), (11) together with $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$, we should obtain the equation in x and y of the pedal.

6. Let (ξ, v) be the Boothian tangential coordinates of the point (x, y) or (p, ϕ) on the pedal; then we have $x = p\cos\phi = p^2\xi$ and $y = p\sin\phi = p^2v$; hence the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes

$$\frac{\xi^2}{a^2} + \frac{v^2}{b^2} = \frac{1}{p^4} = (\xi^2+v^2)^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

which is the central tangential equation of the pedal.

Putting $U \equiv \frac{\xi^2}{a^2} + \frac{v^2}{b^2} - (\xi^2+v^2)^2$, the equation in x and y of the pedal may be otherwise found by eliminating ξ and v from

$$U = 0,\quad \frac{dU}{xd\xi} = \frac{dU}{ydv} = \frac{\xi dU}{d\xi} + \frac{vdU}{dv}. *$$

7. The length of the pedal, estimated from the end of the minor axis is, by putting $\frac{1}{2}\pi - \phi$ for ϕ in (1), found to be

$$\int pd\phi = b\int \frac{d\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}} = bE(e, \phi),$$

where E is an elliptic integral of the first order whose modulus is e, and amplitude ϕ. The entire length of the curve of the pedal is $4bE(e, \frac{1}{2}\pi)$.

8. The polar equation (in r and θ) of the locus of the instantaneous centre, or of the end of the polar subnormal of the ellipse, is, by (1) and (2),

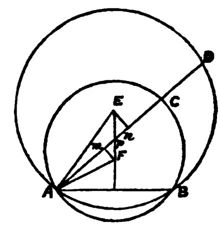
$$r = \frac{e^2b\sin\theta\cos\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

3518. (Proposed by J. J. Walker, M.A.)—Two unequal circumferences meet in AB; find a point C on the arc of the less which lies

* See *Reprint*, Vol. II., pp. 20—22, where, in the Editorial note to the Solution of Quest. 1509, an example of such an elimination is worked out at length. The eliminations indicated in Arts. 5 and 6 are proposed for investigation as Question 3570.

within the other so that, drawing AC and producing it to meet the greater circumference in D, the sum AC + AD may be a maximum.

I. *Solution by the* Rev. G. H. Hopkins, M.A.; H. Murphy; *and others.*



If AC be the radius-vector meeting the line which joins the centres in p, and En, Fm be the perpendiculars from the centres upon it, the problem requires the position of p so that AC + AD may be a maximum.

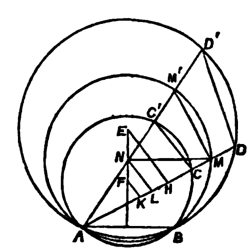
It is obvious that p must lie between the centres; for if it did not, as p moved away from the centres, AC and AD would each be decreasing; but, between the centres, while one of them is increasing the other is decreasing; and the maximum value will be obtained when the position of p is such that the increment of one of the chords AD, AC is equal to the decrement of the other.

The triangles Epn and Fpm are similar, with their angles E and F each equal to the angle pAB.

By whatever angle the angle pAB is increased or diminished by the change of position of p, by equal angles will the angles E and F be increased or diminished.

For an indefinitely small increase of the angle pAB, the increment or decrement of AD will be 2En × the increment of pAB, and the decrement or increment of AC will be 2Fm × the increment of pAB; and when p has the position which makes AC + AD a maximum, these changes in value of AC and AD must be equal; or En is equal to Fm; hence the triangles Enp and Fmp must be equal as well as similar; or Ep must be equal to Fp. Hence the radius-vector AC must bisect the line which joins the centres of the circles; and thus the value of AC can easily be found.

II. *Solution by the* Editor.



Let E, F be the centres of the two circles which intersect in A, B; ACD any common chord through A; and M the middle point of CD. Then, if EH, FK be drawn perpendicular to AD, and HK be bisected in L, L will be the middle point of AM. If, therefore, we draw through L a line LN, at right angles to AD, it will meet EF in its middle point N; hence NM will be equal to NA. We thus obtain the following local theorem:—

If from one of the points of intersection of two circles ABCC′, ABDD′, a chord ACD be drawn, the locus of the middle point of the intercept CD will be a circle ABMM′, whose centre (F) is midway between the centres (E, F) of the two given circles.

We now see at once that AC + AD (= 2AM) will be a maximum when AM is a maximum, that is to say, when AM is the diameter of the circle which is the locus of M, or when the chord takes the position ANC′M′D′, which passes through the middle point N of the line EF joining the centres of the two given circles.

2929. (Proposed by R. TUCKER, M.A.)—Find the locus of the intersection of the lines in Question 2849 (*Reprint*, Vol. XII., p. 23). What does this become when the conic is a circle?

Solution by JAMES DALE.

From a point (f, g, h) straight lines are drawn parallel to the sides of a triangle ABC, and cutting these sides in D, D′; E, E′; F, F′ respectively. Taking ABC as triangle of reference, the trilinear coordinates of the several points are

$$\text{D}, \quad \frac{x}{cf}=\frac{y}{0}=\frac{z}{bg+ch}; \qquad \text{D}', \quad \frac{x}{bf}=\frac{y}{bg+ch}=\frac{z}{0};$$

$$\text{E}, \quad \frac{x}{ch+af}=\frac{y}{ag}=\frac{z}{0}; \qquad \text{E}', \quad \frac{x}{0}=\frac{y}{cg}=\frac{z}{ch+af};$$

$$\text{F}, \quad \frac{x}{0}=\frac{y}{af+bg}=\frac{z}{bh}; \qquad \text{F}', \quad \frac{x}{af+bg}=\frac{y}{0}=\frac{z}{ah};$$

and the condition that the triads DEF, D′E′F′ should be collinear is

$$(af+bg+ch)\left(\frac{gh}{a}+\frac{hf}{b}+\frac{fg}{c}\right)=0;$$

therefore f, g, h lies either at infinity or on the conic

$$\frac{gh}{a}+\frac{hf}{b}+\frac{fg}{c}=0 \quad \text{.............................} \quad (1).$$

The equations to the lines DEF, D′E′F′ are

$$abghx-bh\,(ch+af)\,y+(af+bg)\,(ch+af)\,z=0,$$
$$caghx+(af+bg)\,(ch+af)\,y-cg\,(af+bg)\,z=0,$$

which intersect in the point $\dfrac{x}{af^2}=\dfrac{y}{bg^2}=\dfrac{z}{ch^2}$;

therefore substituting in the equation to the conic, the locus to the point of intersection is $\quad (ax)^{-\frac{1}{2}}+(by)^{-\frac{1}{2}}+(cz)^{-\frac{1}{2}}=0$ .... (2).

This represents a curve of the fourth order and third class, having double points at the angles of the triangle of reference.

When $a=b=c$, the conic (1) becomes the circumscribing circle of the triangle, and (2) becomes $(x)^{-\frac{1}{2}}+(y)^{-\frac{1}{2}}+(z)^{-\frac{1}{2}}=0$, which represents a tricusped hypocycloid having the cusps in the angles A, B, C.

[If an ellipse be inscribed in a triangle so that one focus may always lie on the conic $\left(\dfrac{x}{a}\right)^{\frac{1}{2}}+\left(\dfrac{y}{b}\right)^{\frac{1}{2}}+\left(\dfrac{z}{c}\right)^{\frac{1}{2}}=0$, then the curve (2) is the locus of the other focus. Two solutions of Quest. 2849 are given in p. 23 of Vol. XII. of the *Reprint*.]

3346. (Proposed by C. TAYLOR, M.A.)—An ellipse has double contact with each of two confocal ellipses, of semi-axes a, b, a', b'. Obtain

the relation between the eccentric angles of the points of contact

$$\frac{a^2-a'^2}{aa'}\cos\theta\cos\phi+\frac{b^2-b'^2}{bb'}\sin\theta\sin\phi=0.$$

Solution by F. D. THOMSON, M.A.

If $S=0$ be the equation to the conic having double contact with each of the given conics, these latter must have their equations of the form $S+L^2=0$, $S+M^2=0$, and therefore the chords of contact pass through the point of intersection of a pair of common chords of the two given conics; that is, since they are confocal, through their common centre.

Hence, if $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$ be the confocal conics,

the equation to S is of each of the two forms

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\kappa(y-mx)^2=0,\quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}-1+\kappa'(y-m'x)^2=0;$$

therefore, equating coefficients, we must have

$$\frac{\dfrac{1}{a^2}+\kappa m^2}{\dfrac{1}{a^2+\lambda}+\kappa' m'^2}=\frac{\dfrac{1}{b^2}+\kappa}{\dfrac{1}{b^2+\lambda}+\kappa'}=\frac{m\kappa}{m'\kappa'}-1.$$

Hence, eliminating κ, κ', we get $\quad mm'=-\dfrac{b^2}{a^2}\cdot\dfrac{b^2+\lambda}{a^2+\lambda}$;

or if θ, ϕ be the eccentric angles of the points of contact,

$$m=\frac{b}{a}\tan\theta,\quad m'=\left(\frac{b^2+\lambda}{a^2+\lambda}\right)^{\frac{1}{2}}\tan\phi,$$

therefore $\qquad \tan\theta\tan\phi=-\dfrac{b\,(b^2+\lambda)^{\frac{1}{2}}}{a\,(a^2+\lambda)^{\frac{1}{2}}}$,

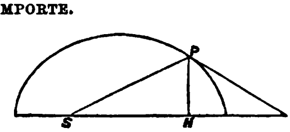
which agrees with the result given in the question.

3303. (Proposed by J. B. SANDERS.)—A body is projected in a direction which makes an angle of 60° with the distance, with a velocity which is to the velocity from infinity as $1:\sqrt{3}$, the force varying inversely as the square of the distance. Find the major axis, the position of the apsis, the eccentricity, and the periodic time.

Solution by N'IMPORTE.

$v^2=\dfrac{2}{3}\dfrac{\mu}{r}$, and $\dfrac{v^2}{\mu}=\dfrac{2}{r}-\dfrac{1}{a}$; therefore

$a=\dfrac{3r}{4}$, $\quad r=\dfrac{4a}{3}$, $\quad 2a-r=\dfrac{2a}{3}$;



or, if S be centre of force, H second focus,

$HP = \frac{1}{2} SP$, and $\angle HPS = 60°$; therefore P is at the end of the further latus rectum, and $SP = a(1+e^2)$; therefore $e = \frac{1}{\sqrt{3}}$, and $\angle PSH = 30°$; also the periodic time $= 2\pi \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} = \pi \frac{3\sqrt{3}\, r^{\frac{3}{2}}}{4\sqrt{\mu}}$.

On Proportion in Geometry. *By* T. S. Aldis, M.A.

1. Ratio is the quantuplicity or how-many-times-ness of one magnitude with respect to another.

2. Proportion is the equality of ratios.

3. To obtain an exact estimate of a ratio, we proceed to find the G. C. M. of the two magnitudes. The successive remainders are the differences of multiples of the magnitudes, and the numbers expressing these multiples give us a continual approximation to the true value of the ratio, though if the magnitudes be incommensurable we can never exactly express it.

4. A, B, C, D will therefore be proportional when the process of finding the G. C. M. of A and B is parallel to that for finding the G. C. M. of C and D.

Theorem I.—Parallelograms of the same altitude are proportional to their bases.—*Proof:* The process of finding the G. C. M. of the bases is identical with that for finding the G. C. M. of the areas.

Theorem II.—In equal circles, angles at the centre are proportional to the arcs on which they stand.—*Proof:* As above.

Theorem III.—Straight lines are cut proportionally by parallel straight lines. The proof in the same way.

5. Obviously, ratios that are equal to the same ratio, are equal to each other.

6. Also, if $mA >=< nB$ as $mC >=< nD$, the processes for finding the G. C. M. of A and B, and also of C and D, will run on parallel.

Theorem IV.—If $A : B = C : D$, then $mA : mB = nC : nD$.

Theorem V.—Equals cannot have the same ratio to unequals.

Theorem VI.—A cannot have to B the ratio that a less than A has to a greater than B. (V. and VI. need not be proved, but assumed with *beginners.*)

Theorem VII.—If $A : B = C : D$, then $A : C :: B : D$.

For $mA : mB = nC : nD$, therefore, by V. and VI., $mA >=< nC$, as $mB >=< nD$; therefore, by (6), $A : C = B : D$.

This Theorem is, as every one who has examined the subject knows, the great difficulty in a strict Geometrical treatment of proportion.

Other Theorems can now be readily deduced, as

$$mA \pm nB : A = mC \pm nD : C.$$

We can now prove the converse of Theorem III. Thus the similarity of triangles is mastered, and the rest of the subject presents no great difficulty.

The writer believes that in this way the subject of Proportion is made exact as well as simple. Theorems V. and VI. are the hardest points, but, under this aspect, are by no means difficult to prove.

3536. (Proposed by Professor CAYLEY.)—A particle describes an ellipse under the simultaneous action of given central forces, each varying as (distance)$^{-2}$, at the two foci respectively: find the differential relation between the time and the excentric anomaly.

Solution by the Rev. R. TOWNSEND, M.A., F.R.S.

Employing the usual notation, since, on elementary principles of motion,

$$v^2 = \frac{1}{a}\left(\mu\frac{r'}{r} + \mu'\frac{r}{r'}\right);$$

since, again, $x = a\cos u$, $y = b\sin u$, so that

$$v^2 = \frac{dx^2+dy^2}{dt^2} = (a^2\sin^2 u + b^2\cos^2 u)\frac{du^2}{dt^2} = a^2(1-e^2\cos^2 u)\frac{du^2}{dt^2};$$

and since, finally, $r = a(1+e\cos u)$, $r' = a(1-e\cos u)$, therefore at once

$$\left(\frac{du}{dt}\right)^2 = \frac{1}{a^3}\left(\frac{\mu}{(1+e\cos u)^2} + \frac{\mu'}{(1-e\cos u)^2}\right),$$

which is consequently the required relation.

3474. (Proposed by the EDITOR.)—Given two sides of a triangle, and suppose (I) the included angle to vary uniformly, (II) the third side to vary uniformly; find (1) the average area of the inscribed circle, (2) the mean value of the ratio of the circumscribed to the inscribed circle, (3) the minimum value of the circumscribed circle, (4) the maximum value of the inscribed circle; also (III) find what these values become when the two given sides are equal.

Solution by STEPHEN WATSON.

Let a and b be the given sides ($a > b$), 2θ the included angle, and x the third side; then $x^2 = a^2 + b^2 - 2ab\cos 2\theta = (a+b)^2 - 4ab\cos^2\theta$.

I. (1.) Here the limits are θ from 0 to $\frac{1}{2}\pi$, and the measure of the number of circles formed is π; hence the average area of the inscribed circle is

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \pi r^2 d(2\theta) = \frac{1}{2}\int_0^{\frac{1}{2}\pi}(a+b-x)^2\tan^2\theta\,d\theta, \text{ or putting } m^2 = \frac{4ab}{(a+b)^2}$$

$$= \int_0^{\frac{1}{2}\pi}\left\{(a+b)^2 - 2ab\cos^2\theta - (a+b)^2(1-m^2\cos^2\theta)^{\frac{1}{2}}\right\}\tan^2\theta\,d\theta,$$

$$= \int_0^{\frac{1}{2}\pi}\left\{-2ab\cos^2\theta + \tfrac{1}{2}(a+b)^2\left(m^2\cos^2\theta + \frac{1}{2^2}m^4\cos^4\theta + \frac{1.3}{2^3.3}m^6\cos^6\theta\right.\right.$$

$$\left.\left. + \frac{1.3.5}{2^4.3.4}m^8\cos^8\theta + \&c.\right)\right\}\tan^2\theta\,d\theta$$

$$= -\tfrac{1}{2}ab\pi + \tfrac{1}{8}(a+b)^2\pi\left(m^2 + \frac{1}{2^4}m^4 + \frac{3^2}{2^6.3^2}m^6 + \frac{3^2.5^2}{2^8.3^2.4^2}m^8 + \&c.\right).$$

I. (2.) The mean value of the ratio is

$$\frac{1}{\pi}\int \frac{r^2}{R^2}\,d(2\theta) = \frac{2}{\pi}\int\left\{\tfrac{1}{4}(a+b-x)^2\tan^2\theta \div \frac{x^2}{4\sin^2 2\theta}\right\}d\theta$$

$$= \frac{8}{\pi}\int_0^{\frac{1}{2}\pi} d\theta\,\frac{(a+b-x)^2\sin^4\theta}{x^2}$$

$$= \frac{8}{\pi}\int_0^{\frac{1}{2}\pi} d\theta\left\{1 - \frac{2(a+b)}{x} + \frac{(a+b)^2}{(a+b)^2-4ab\cos^2\theta}\right\}\sin^4\theta$$

$$= \frac{6a^2+2ab-b^2}{2a^2} - \frac{16}{\pi}\int_0^{\frac{1}{2}\pi}(1-m^2\cos^2\theta)^{-}\ \sin^4\theta\,d\theta$$

$$= \frac{b(2a-b)}{2a^2} - \frac{3}{4}\left(\tfrac{1}{3}m^2 + \frac{3^2}{2^3.3.4}m^4 + \frac{3^2.5^2}{2^5.3^2.4.5}m^6 + \&c.\right).$$

(3.) The minimum value will plainly be that when the diameter coincides with the longer side a, and it is equal to $\frac{1}{4}\pi a^2$.

(4.) Put $a+b = u$, $s = \frac{1}{2}(u+x)$; then we have to find

$$r^2 = \frac{(s-a)(s-b)(s-x)}{s} = -s^2 + 2us - (ab+u^2) + \frac{abu}{s} = \text{max.};$$

therefore $$s^3 - us^2 + \tfrac{1}{2}abu = 0,$$

and from this equation s can be found by the solution of a cubic. Denote the result by s_1; then the area of the maximum inscribed circle is

$$\pi\,(s_1-a)(s_1-b)(u-s_1)\,s_1^{-1}.$$

II. (1.) Here x may vary from $a-b\,(=u_1)$ to $a+b\,(=u)$, and the number of its variations is $2b$; hence the average area of the inscribed circle is

$$\frac{1}{2b}\int_{u_1}^{u}\pi r^2\,dx = \frac{\pi}{b}\int_a^u\left\{-s^2+2us-(ab+u^2) + \frac{abu}{s}\right\}ds$$

$$= \pi\left\{au\log\frac{u}{a} - b\,(a+\tfrac{1}{3}b)\right\}.$$

II. (2.) The mean value of the ratio here is

$$\frac{1}{2b}\int_{u_1}^{u} dx\,\frac{r^2}{R^2} = \frac{8}{a^2b^3}\int_{u_1}^{u} dx\,\frac{(s-a)^2(s-b)^2(s-x)^2}{x^2}$$

$$= \frac{1}{8a^2b^2}\int_{u_1}^{u} dx\,\left\{x^4 - 2ux^3 + (u^2-2u_1^2)\,x^2 + 4uu_1^2x + u_1^2(u_1^2-2u^2) \right.$$
$$\left. - 2uu_1^4x^{-1} + u^2u_1^4x^{-2}\right\}$$

$$= \frac{1}{a^2b^3}\left\{\tfrac{1}{240}u^5 - \tfrac{1}{12}u^3u_1^2 + \tfrac{1}{3}u^2u_1^3 - \tfrac{3}{16}uu_1^4 - \tfrac{1}{15}u_1^5 - \tfrac{1}{4}uu_1^4\log\frac{u}{u_1}\right\}.$$

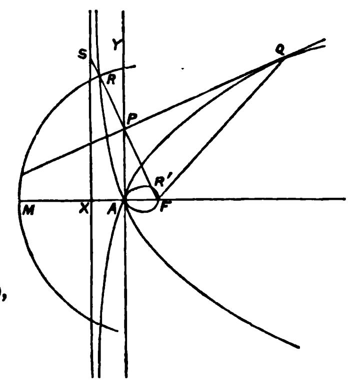
III. When $b=a$ the values become

I. (1.) $= (8-\frac{5}{3}\pi)\,a^2 = (\cdot 046)\,\pi a^2$, I. (2.) $= \dfrac{7}{2} - \dfrac{32}{3\pi} = \cdot 105$;

II. (1.) $= (\log 4 - \frac{4}{3})\,\pi a^2 = (\cdot 053)\,\pi a^2$, II. (2.) $= \frac{2}{15} = \cdot 133$;
(3.) $= \frac{1}{4}\pi a^2 = (\cdot 25)\,\pi a^2$, (4.) $= \frac{1}{2}(5\sqrt{5}-11)\,\pi a^2 = (\cdot 09)\,\pi a^2$.

3508. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—A tangent is drawn to a parabola at a point Q; a perpendicular, making an angle θ with the axis, is drawn to it from the focus, meeting the tangent in the point P; and the difference between the parabolic arc AQ and the portion of the tangent QP is equal to the focal distance of the vertex AF. Prove (1) that $\sec\theta + \tan\theta = e$, e being the Napierian base; and (2) find geometrically the position of the point Q.

Solution by the REV. G. H. HOPKINS, M.A.; A. B. EVANS, M.A.; S. WATSON; R. W. GENESE, B.A.; G. S. CARR; *and others.*



1. By properties of the parabola, the angles QFP, PFA, QPY are equal; therefore

$$PQ = AF \tan\theta$$

The intrinsic equation to the parabola with AY as the initial line is

$$s = 2AF\int \sec^3\phi \, d\phi;$$

therefore AQ − PQ

$$= 2AF\int_0^\theta \sec^3\phi\, d\phi - AF\tan\theta\sec\theta$$

$= AF.\log(\tan\theta + \sec\theta) = AF \ldots (1)$, by the problem;

therefore $\log(\tan\theta + \sec\theta) = 1$,

or $\tan\theta + \sec\theta = e$.

2. Draw the parabola, and also the curve represented by

$$r = a(\sec\theta \pm \tan\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where F is the pole, $AF = AX = a$, and $\angle PFA = \theta$. Draw any radius vector FR′R to the curve, meeting the tangent AY at the vertex of the parabola in P; then $PR = PR' = a\tan\theta$. Through P draw PQ at right angles to RR′, to meet the parabola in Q; then PQ is easily seen to be a tangent to the parabola. Now $FR = AF(\sec\theta + \tan\theta) = e.AF$, if Q be the position of the point according to the problem. Hence on the axis of the parabola take $FM = e.AF$; from centre F with radius FM draw a circle cutting the curve in R; join FR, which cuts the curve again in R′; bisect RR′ in P; draw PQ at right angles to FP to meet the parabola in Q: then Q is such a point that arc AQ − PQ = AF.

[Mr. HOPKINS's equation (1) may be otherwise obtained thus:—

$$AF = AQ - QP = \int FP.\,d\theta = AF\int \sec\theta\, d\theta = \log(\tan\theta + \sec\theta).$$

Equation (2) is that of the *Logocyclic Curve*, by the aid of which Dr. BOOTH has succeeded in exhibiting the whole theory of logarithms in a simple and beautiful geometrical form, first in a paper printed in the *Proceedings of the Royal Society* for June, 1858, and afterwards in the *Quarterly Journal of Mathematics* for 1860, in an article which M. CHASLES (p. 145 of his recent *Rapport sur les Progrès de la Géométrie*) calls "son très-remarquable mémoire sur la courbe qu'il a appelée Logocyclique."]

3515. (Proposed by H. McColl.)—If P and Q be random points within a circle, show that the chance, that the circle of which P is the centre and PQ the radius will lie wholly within the given circle, is $\frac{1}{6}$.

Solution by G. S. Carr; *the* Proposer; S. Watson; *and others.*

The chance that P falls within a ring whose bounding radii are x and $x+dx$ is $2xdx$, the radius of the given circle being taken as our unit. Given that P falls at a distance x from the centre of the given circle, the necessary conditions of the question will be satisfied when (and only when) Q falls inside the circle whose centre is P and radius $1-x$; and the chance of this is $\pi(1-x)^2 \div \pi = (1-x)^2 =$ the chance for all positions of P in the infinitesimal ring already spoken of. The chance therefore that the point P falls on this ring, and that also the necessary conditions of the question will be satisfied, is $(1-x)^2 . 2xdx$. Conceiving the circle to be made up of an infinite number of such infinitesimal rings, the required chance is evidently $\int_0^1 2x(1-x)^2 dx = \int_0^1 2(1-x)x^2 dx = \frac{1}{6}$.

3520. (Proposed by J. W. L. Glaisher, B.A.)—Prove that

$$\left(\frac{d}{dq}\right)^{2i} e^{\frac{q^2}{p^2}} = p\left(-\frac{2d}{p\,dp}\right)^i \frac{e^{\frac{q^2}{p^2}}}{p}.$$

Solution by the Proposer.

From the known integral $\int_{-\infty}^{\infty} e^{-p_1x^2+qx}\,dx = e^{\frac{q^2}{4p_1}}\left(\frac{\pi}{p_1}\right)^{\frac{1}{2}}$,

we obtain
$$\left(-\frac{d}{dp_1}\right)^i \frac{e^{\frac{q^2}{4p_1}}}{\sqrt{p_1}} = \left(\frac{d}{dq}\right)^{2i} \frac{e^{\frac{q^2}{4p_1}}}{\sqrt{p_1}}.$$

Put $4p_1 = p^2$, therefore $2dp_1 = p\,dp$, therefore $\dfrac{d}{dp_1} = \dfrac{2d}{p\,dp}$, and the theorem follows at once.

3522. (Proposed by T. T. Wilkinson, F.R.A.S.)—Let ABC be a triangle; AE, BF, CG the lines bisecting the interior angles; and AE′, BF′, CG′ those bisecting the exterior angles; then circles drawn on FF′, GG′, EE′ as diameters have the same radical axis.

I. *Solution by* R. W. GENESE, B.A.

The circle on EE′ as diameter is the locus of a point P such that the ratio BP : CP = $c : b$. Let this circle meet the circle on FF′ as diameter in the points X, Y; then

BX : CX = $c : b$, CX : AX = $a : c$, therefore BX : AX = $a : b$.

Hence X is a point on the circle on GG′. Similarly Y may be shown to be on this circle. Thus the three circles pass through the same two points, or have a common radical axis.

II. *Solution by* STEPHEN WATSON.

The trilinear equations of the circles on EE′, FF′, GG′ are

$$\left.\begin{array}{l} 2\cos B\,.\,\gamma\alpha - 2\cos C\,.\,\alpha\beta = \beta^2-\gamma^2 \\ 2\cos C\,.\,\alpha\beta - 2\cos A\,.\,\beta\gamma = \gamma^2-\alpha^2 \\ 2\cos A\,.\,\beta\gamma - 2\cos B\,.\,\gamma\alpha = \alpha^2-\beta^2 \end{array}\right\} \text{........................}(1);$$

for the condition that the lines $\alpha = r(\beta-\gamma)$ and $\alpha = r_1(\beta+\gamma)$ shall be at right angles is $(r-r_1)\cos B-(r+r_1)\cos C = 1$;

whence, substituting the values of r, r_1, we have the first of (1). Since any one of the equations (1) is the sequence of the other two, the three circles must pass through the same two points, and therefore have the same radical axis. Their centres must also lie in a right line. This line is

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=0 \text{..................................}(2),$$

for the coordinates of the middle points of E, E′ are $0,\ \dfrac{2b\Delta}{b^2-c^2},\ -\dfrac{2c\Delta}{b^2-c^2}$; hence the line joining A to this point is $c\beta+b\gamma=0$. Similarly, $a\gamma+c\alpha=0$, $b\alpha+a\beta=0$ are the lines joining B, C to the middles of FF′, GG′; and (2) follows at once from these.

3554. (Proposed by T. MITCHESON, F.E.I.S.)—A beam of uniform thickness, and weight W, is placed with one end on a horizontal and the other on an inclined plane; if α and β be the respective angles at which the plane and the beam are inclined to the horizontal plane, and P the horizontal force applied at the foot of the beam in order to maintain it in equilibrium, prove that $W = 2P(\cot\alpha+\tan\beta)$.

Solution by Q. W. KING; J. A. SLADE; R. TUCKER, M.A.; *the* PROPOSER; *and others.*

The beam being uniform, its weight acts through its middle point. Take moments about the point of contact with the horizontal plane; then, putting R for the reaction of the plane at the upper end, we have

$$W\cos\beta = 2R\cos(\alpha-\beta) \text{ .... } (1).$$

Resolving horizontally, we get $P = R\sin\alpha$ .... (2).

From (1) and (2) we readily obtain the given relation.

3434. (Proposed by T. COTTERILL, M.A.)—Show that

$$ax^3(y^2-z^2)+by^3(z^2-x^2)+cz^3(x^2-y^2)=0$$

(a, b, c variable parameters) is the equation to a system of quintics having 22 points in common, and such that any two of the curves intersect again in 3 points lying on a conic circumscribing the triangle of reference. If a quintic of the system is fixed, the conics thus determined by the other quintics pass through a fixed point on the fixed quintic.

Solution by the PROPOSER.

The system of quintics in question form a reseau passing through the 4 points $(1, \pm 1, \pm 1)$ and having a node at each vertex of the triangle of reference with common tangents; viz., the pair of lines through it and the 4 points given above. All such quintics have therefore $(4+6\times 3)=22$ common points, and two of the quintics will intersect again in 3 points. Let $a'x^3(y^2-z^2)+b'y^3(z^2-x^2)+c'z^3(x^2-y^2)=0$ be another of the quintics, and $A=bc'-cb'$, $B=ca'-ac'$, $C=ab'-ba'$. Hence, for the remaining

common points $\quad \dfrac{x^3(y^2-z^2)}{A}=\dfrac{y^3(z^2-x^2)}{B}=\dfrac{z^3(x^2-y^2)}{C}$

lying on the conic $Ayz+Bzx+Cxy=0$, circumscribing the triangle of reference, and passing through the points $\left(\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}\right)$ and $\left(\dfrac{1}{a'}, \dfrac{1}{b'}, \dfrac{1}{c'}\right)$.

The same results can also be obtained by observing that the quintics are the inverts by the quadric inversion $xx'=yy'=zz'$ of the reseau of quartics given by the equation

$$ayz(y^2-z^2)+bzx(z^2-x^2)+cxy(x^2-y^2)=0,$$

which evidently pass through 13 common points. Two such quartics intersect again in 3 points which, proceeding as before, can be proved to be collinear. This remarkable system of quartics is fully discussed by SARDI, in Battaglini's *Giornale di Matematiche* for 1868, p. 217.

3526. (Proposed by S. BILLS.)—To find two positive cube numbers besides 8 and 27, such that their sum shall be 35.

I. *Solution by the* PROPOSER; DR. HART; *and others.*

Let x and y denote the roots of the required cubes, then we have to find $x^3+y^3=35$.

Assume $x=p+q$ and $y=p-q$; then the above becomes $2p^3+6pq^2=35$; whence $q^2=\dfrac{35-2p^3}{6p}$; therefore $6p(35-2p^3)$ must be a square.

Assume $p=\frac{5}{2}+r$; then by substitution, and dividing by 9, we must have

$$\tfrac{25}{4}-60r-50r^2-\tfrac{40}{3}r^3-\tfrac{4}{3}r^4=\square=\left(\tfrac{5}{2}-12r-\tfrac{124}{5}r^2\right)^2, \text{ suppose.}$$

Developing this, we find $r=-\frac{1265}{2018}$, whence $p=\frac{5}{2}+r=\frac{5}{2}-\frac{1265}{2018}=\frac{1890}{1009}$;

also, $\qquad q=\left(\dfrac{35-2p^3}{6p}\right)^{\frac{1}{2}}=\dfrac{25327}{18.1009}$;

whence we obtain $x=p+q=\frac{59347}{18162}$, and $y=\frac{8693}{18162}$.

II. *Solution by* A. B. Evans, M.A.; Rev. U. J. Knisely; *and others.*

Let x^3 and y^3 represent the two required cube numbers; then

$$x^3+y^3 = 35 \text{ .... } (1).$$

Assume $x+y=5n$; then from (1) we find $(x-y)^2 = \dfrac{84n-75n^4}{9n^2}$; hence $84n-75n^4$ must be a square. Since $n=1$ satisfies the condition $84n-75n^4 = \square$, assume $n=1-m$, and let

$$84(1-m)-75(1-m)^4 = 9+216m-450m^2+300m^3-75m^4 = (3+36m-rm^2)^2;$$

then $\qquad (1746-6r)\,m^2-(300+72r)\,m^3+(r^2+75)m^4 = 0$ ....(2).

Let $6r = 1746$, or $r=291$; then from (2)

$$m = \frac{300+72r}{r^2+75} = \frac{21252}{84756} = \frac{253}{1009}, \text{ and } n = 1-m = \frac{756}{1009}.$$

Also, $x-y = \dfrac{1}{3n}\sqrt{(84n-75n^4)} = \dfrac{25327}{9081}$, and $x+y=5n=\dfrac{3780}{1009}$;

therefore $\qquad x = \dfrac{59347}{18162},\ y = \dfrac{8693}{18162}$; and

$$x^3 = \left(\frac{59347}{18162}\right)^3 = \frac{209024075174923}{5990885427528},\quad y^3 = \left(\frac{8693}{18162}\right)^3 = \frac{656914788557}{5990885427528}.$$

3433. (Proposed by Hugh McColl.)—ABCD, Abcd are two squares, having the sides AB, Ab coinciding in direction, and also AD, Ad; likewise Ab=½AB and Ad=½AD. Through a random point in the square Abcd a random line is drawn. Show that ·29557, ·14570, ·55873 are the respective decimal approximations to the probabilities of the three following mutually exclusive events: (1) that the random line cuts AB and AD; (2) that it cuts some other pair of adjacent sides of the square ABCD; (3) that it cuts opposite sides of it.

Solution by the Proposer.

The analytical enunciation of the first part of the problem is this:—

Find the probability that the three events $\tfrac{1}{2}\pi > \theta > 0$, $2 > y + x\tan\theta$, $2 > x + y\cot\theta$ will simultaneously happen; all values of θ between 0 and π, of x between 0 and 1, and of y between 0 and 1, being equally probable, while all other values of these variables are excluded.

Employing the same notation as in my solution of Quest. 3440, we have

$$p_1 = p(1.2.3) = p(1.2.4)+p(1.3:4),$$

for 2 . 4 implies 3, and 3 : 4 implies 2,

$$= p(5.2)+p(6.3) = 2p(5.2), \text{ for } p(5.2) = p(6.3),$$

as may be seen at once by exchanging x and y, and also $\tan\theta$ and $\cot\theta$;

$$p(5.2) = p(7.5)+p(:7.5.2) = p(7.5)+p(:7.5.2.8),$$

for 2 implies 8,

$$= p(7.5)+p(:7.5.2.9)+p(:7.5.2.8:9)$$
$$= p(7.5)+p(:7.10.2)+p(:7.2.8.11).$$

TABLE OF REFERENCE.

1 ...... $\frac{1}{2}\pi > \theta > 0$	7_2 ...... $y_1 > 1 = \cot\theta > x$
2 ...... $2-x\tan\theta > y = y_1 > y$ say	8 ...... $y_1 > 0 = 2\cot\theta > x$
3 ...... $(2-x)\tan\theta > y = y_2 > y$ say	9_8 ...... $2\cot\theta > 1 = 2 > \tan\theta$
4 ...... $y_2 > y_1 = \tan\theta > 1$	10 ...... $\tan^{-1}2 > \theta > \frac{1}{4}\pi = 5.9$
5 ...... $\frac{1}{2}\pi > \theta > \frac{1}{4}\pi = 1.4$	11 ...... $\frac{1}{2}\pi > \theta > \tan^{-1}2 = 5:9$.
6 ...... $\frac{1}{4}\pi > \theta > 0 = 1:4$	

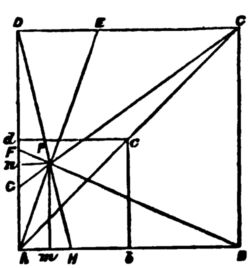
For $p\,(7.5)$ we have the integral $\frac{1}{\pi}\iint d\theta\,dx$, the limits being $\frac{1}{2}\pi > \theta > \frac{1}{4}\pi$, $\cot\theta > x > 0$; and for each of the other two, we have the integral $\frac{1}{\pi}\iiint d\theta\,dx\,dy$, the limits in the one case being $\tan^{-1}2 > \theta > \frac{1}{4}\pi$, $1 > x > \cot\theta$, $y_1 > y > 0$, and in the other $\frac{1}{2}\pi > \theta > \tan^{-1}2$, $2\cot\theta > x > \cot\theta$, $y_1 > y > 0$. These integrations are easy and elementary, and we readily obtain

$$p_1 = \frac{4}{\pi}\left(\cot^{-1}3 - \log 2 + \tfrac{5}{8}\log 5\right) = \cdot 29557.$$

The conditions to be satisfied in p_3 are evidently equivalent to those of Question 3385, of which a Solution has already been given (see the *Educational Times* for May), so that

$$p_3 = 1 - \frac{2}{\pi}\log 2 = \cdot 55873 \text{ and } p_2 = 1 - p_1 - p_3 = \cdot 14570.$$

II. *Solution by* STEPHEN WATSON.



Let P be the random point, through which draw BF, DH, AE, CG, meeting AD, AB, CD, DA respectively in F, H, E, G; also draw Pm perpendicular to AB, and Pn to AD. Put AB$=2a$, A$m=ax$, A$n=ay$; then the total number of positions of the point and line is $a^2\pi$. When the line through P intersects AB and AD, the number of its positions is expressed by

$$\angle \text{FPD} = n\text{PD} - n\text{PF} = \tan^{-1}\frac{2-y}{x} - \tan^{-1}\frac{y}{2-x} \;\dots\,(4).$$

Hence the chance in case (1) is

$$p_1 = \frac{1}{a^2\pi}\int_0^1 d(ax)\int_0^1 (4)\,d(ay) = \frac{1}{\pi}\int_0^1 dx\left\{2\tan^{-1}\frac{2}{x} - \tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{2-x} + \tfrac{1}{2}x\log\frac{1+x^2}{4+x^2} + \tfrac{1}{2}(2-x)\log\frac{(2-x)^2+1}{(2-x)^2}\right\}$$

$$= \frac{4}{\pi}\left(\cot^{-1}3 - \log 2 + \tfrac{5}{8}\log 5\right) = \cdot 29557.$$

The pair of adjacent sides BC, CD cannot be cut by any of the random lines. In order that the adjacent pair AD, DC may be cut, the point P must lie in the triangle Acd, and the random line within the angle

$$\text{APG} = \text{AP}n - \text{GP}n = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{2-y}{2-x}\;\dots\dots\dots\dots\,(6).$$

Hence doubling because the adjacent pair AB, BC may be cut, we have for the chance in case (2),

$$p_2 = \frac{2}{a^2\pi}\int_0^1 d(ax)\int_x^1 (6)\, d(ay) = \frac{2}{\pi}\int_0^1 dx\,\Big\{\tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{2-x}$$

$$-\tfrac{1}{2}\pi - \tfrac{1}{2}x\log\frac{1+x^2}{2x^2} - \tfrac{1}{2}(2-x)\log\frac{(2-x)^2+1}{2(2-x)^2}\Big\}$$

$$= \frac{4}{\pi}\left(\tfrac{3}{2}\log 2 - \tfrac{3}{8}\log 5 - \cot^{-1} 3\right) = \cdot 14570.$$

Hence the chance in (3) is $p_3 = 1 - p_1 - p_2 = 1 - \dfrac{1}{\pi}\log 2 = \cdot 55873.$

3567. (Proposed by Dr. Hirst, F.R.S.)—Given any three fixed straight lines l, m, n, and any three fixed collinear points L, M, N. If, from a variable point in l, lines be drawn through M and N to cut m and n, the envelope of the conic which passes through the four points of intersection, as well as through L, will be another conic touching m and n where the latter lines are intersected by l.

3509. (Proposed by Dr. Hirst, F.R.S.)—Given three concurrent lines l, m, n and a fixed point L. The envelope of a conic which touches l and has a pair of conjugate foci situated on m and n and in line with L, is a circle whose centre is L.

Solutions by the Proposer; S. Watson; J. J. Walker, M.A.; *and others.*

The intersection of m and n in Quest. 3567 being taken as one of the vertices of the triangle of reference, M and N being the two others, the lines l, m, n will have equations of the following forms:

$$l \equiv ax + by + cz = 0, \quad m \equiv b_1 y + c_1 z = 0, \quad n \equiv b_2 y + c_2 z = 0.$$

Again, P being the point in which l is intersected by the variable line $\beta y + \gamma z = 0$, the equations of $\overline{\mathrm{PM}}$ and $\overline{\mathrm{PN}}$ are easily seen to be, respectively,

$$\beta a x + (\beta c - \gamma b)\, z = 0, \quad \gamma a x - (\beta c - \gamma b)\, y = 0;$$

and the equation of the conic, through the four points in which $\overline{\mathrm{PM}}$, $\overline{\mathrm{PN}}$ intersect m and n, whose envelope is required, will be

$$[\beta a x + (\beta c - \gamma b)\, z]\,[\gamma a x - (\beta c - \gamma b)\, y] + \lambda(\beta c - \gamma b)^2 mn = 0 \;\ldots\ldots\; (1),$$

provided an appropriate value be given to the parameter λ.

Putting $x = 0$, the intersections of this conic and the line $\overline{\mathrm{MN}}$ are found to satisfy the equation $\qquad \lambda mn = yz \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2),$

which, it will be observed, does not contain either β or γ. By substituting in (2), therefore, the coordinates of the given point L, λ is determined; and for that value of λ all the conics represented by (1) will not only pass through L, but likewise through the second (fixed) point L′ on $\overline{\mathrm{MN}}$, whose coordinates satisfy (2). The equation (1) may be written in the form

$$A\beta^2 + B\beta\gamma + C\gamma^2 = 0,$$

where $\quad A = c\,[\lambda cmn - (ax + cz)\,y], \quad B = alx + 2bcyz - 2\lambda bcmn,$

$$C = b\,[\lambda bmn - (ax + by)\,z]\,;$$

whence we deduce, for the equation of the required envelope, $4AC - B^2 = 0$; which on reduction becomes $a^2x^2\,(4\lambda bcmn - l^2) = 0$,
and represents, irrespective of the point-pair (L, L′), a conic which touches m and n where these lines are intersected by l.

The reciprocal of the theorem just proved may be thus enunciated:

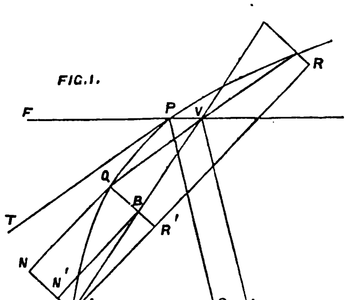
Given any three fixed points L, M, N, and any three fixed concurrent lines l, m, n. If the intersections with m and n of a variable line through L be connected with the fixed points M and N, the conic which touches the four connectors, as well as the line l, will not only touch another fixed line l' (concurrent with l, m, n), but likewise envelope a conic which touches LM and LN at the points M and N.

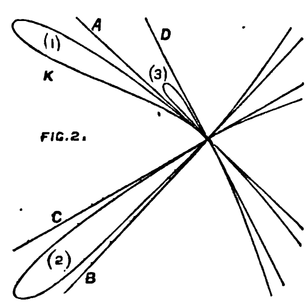
If M and N be the circular points at infinity, this theorem reduces itself to the one given in Question 3509. I may mention that I am at present in possession of a method of transformation which is very fruitful in theorems concerning envelopes. By it all such theorems are shown to be consequences of the generation of a right line by a moving point, or of a point by a revolving right line.

3488. (Proposed by the Editor.)—A circle is drawn on a double ordinate of a parabola as diameter, and the ends of this diameter are joined with the points in which the circle is cut by a line drawn through its centre and the vertex of the parabola: trace the locus of the feet of the perpendiculars from the vertex of the parabola on the four joining lines.

Solution by G. S. Carr.

Let fig. (1) represent the conditions of the question, A being the vertex of the parabola, S the focus, FPV the diameter to which the ordinates are taken, and R, R′, N, N′ the feet of the perpendiculars in the question.





Put $AS = a$, $\angle TPF = \alpha$, and therefore $PSL = 2\alpha$, $\angle VAL = \phi$; and let the polar coordinates of the locus be $r\,(= AR$ or $AR')$ and $\theta\,(= \angle RAL)$.

Draw VL parallel to PS; then we have

$$\theta = \phi - \mathrm{VAR} = \phi - \tfrac{1}{2}(\phi - a) = \tfrac{1}{2}(\phi + a), \quad \text{therefore} \quad \phi = 2\theta - a \;\ldots (1);$$

$$r = (\mathrm{AV} \pm \mathrm{BV}) \cos \mathrm{VAR} \ldots\ldots (2),$$

$$\mathrm{AV} = \mathrm{VL}\frac{\sin 2a}{\sin\phi} = \frac{a\sin 2a}{\sin^2 a \sin\phi} = \frac{2a\cot a}{\sin(2\theta - a)}, \text{ by } (1);$$

$$\mathrm{BV}^2 = \mathrm{QV}^2 = 4\mathrm{SP}.\mathrm{PV} = 4\mathrm{SP}(\mathrm{AL}-\mathrm{AS}),$$

$$\mathrm{AL} = \mathrm{VL}\frac{\sin(2a-\phi)}{\sin\phi}, \quad \mathrm{SP} = \frac{a}{\sin^2 a};$$

$$\mathrm{BV}^2 = 4\frac{a}{\sin^2 a}\left(\frac{a\sin(2a-\phi)}{\sin^2 a\sin\phi} - a\right) = \frac{4a^2}{\sin^2 a}\left\{2\cot(2\theta - a) - \cot a\right\}\cot a.$$

Hence, by (2), the polar equation of the locus of R and R′ is

$$r = \frac{2a\cos(\theta - a)}{\sin a}\left[\frac{\cos a}{\sin(2\theta - a)} \pm \left\{\cot a\,[2\cot(2\theta - a) - \cot a]\right\}^{\frac{1}{2}}\right] \ldots (3).$$

For the locus of the other two points (N, N′) let (r', θ') be the coordinates; then AN or AN′ $= r' = (\mathrm{AV} \pm \mathrm{BV})\sin \mathrm{VAR}$, and $\theta = \theta' - \tfrac{1}{2}\pi$; hence the equation in r', θ' becomes identical with (3), which therefore represents the locus of all four points.

To find the tangents at the origin, put $r=0$ in (3); thus $\cos(\theta - a) = 0$, and therefore $\theta = a + \tfrac{1}{2}\pi$ .... (A),
or else the second factor of (3) vanishes, which gives

$$\cot(2\theta - a) = \cot a \text{ or } \cot a + 2\tan a.$$

From the first value $\theta = a$ .... (B).
From the second $\cot 2\theta = \tfrac{1}{3}\cot a$, whence $\theta = \tfrac{1}{2}\cot^{-1}(\tfrac{1}{3}\cot a) + \tfrac{1}{2}\pi$ ...(C),
or $\theta = \tfrac{1}{2}\cot^{-1}(\tfrac{1}{3}\cot a)$ .... (D);
A, B, C, D are the equations to the four tangents at the origin.

The limits of θ are given by $\tan(2\theta - a) = 2\tan a$ at P, and $2\theta - a = 0$ at infinity.

An inspection of (3) shews that, as θ diminishes, the two values of r vanish simultaneously when $\theta = a + \tfrac{1}{2}\pi$, both values being previously positive, but in passing through zero one of them changes sign while the other does not, but increases positively. Thus (A) is a common tangent to three branches at the origin, two of which form a cusp of the first species.

The part of the curve near the origin is shown in figure (2), and is formed in the following manner:—As the centre of the generating circle moves from P to infinity, the points R, R′ start together from a point near P, and N, N′ from K. R moves off to infinity. R′, N, N′ reach the origin together, N, N′ having formed loop (1). N′ passes on to infinity, while R′ and N form loops (2) and (3), and passing through the origin together both points describe infinite branches.

Figure (2) is sketched from an accurate plan of the curve made with a latus rectum of 26 inches, and the diameter through its extremity. The sketch is, of necessity, somewhat incorrect. The true dimensions of the loops, in parts of the latus rectum, are about as follows:—(1) and (2), length $= \frac{1}{9}$, width $= \frac{1}{130}$; (3), length $= \frac{1}{50}$, width $= \frac{1}{300}$. The order of contact of the curves at the origin and the maximum value of r in the loops may be determined, *secundum artem*, but the labour of solving the equations $\dfrac{dr}{d\theta} = 0$, &c., would scarcely be repaid.

3460. (Proposed by J. HOPKINSON, D.Sc., B.A.)—There are $a+b$ balls in a bag, a white and b black, a being greater than b. A ball is to be drawn out at random, and some one offers any even bet that it is black; what part of one's fortune should one lay on the event, assuming Bernoulli's theory of value of expectation?

Solution by MILLICENT COLQUHOUN.

Suppose the xth part of one's fortune be laid on white, the expectation and loss are

$$\frac{ax}{a+b}\int_1^{1+x}\frac{dx}{x},\qquad \frac{bx}{a+b}\int_1^{1-x}\frac{dx}{x};$$

and the sum of these should be zero; therefore

$$a\int_1^{1+x}\frac{dx}{x}+b\int_1^{1-x}\frac{dx}{x}=0,\quad\text{or}\quad a\log(1+x)+b\log(1-x)=0;$$

therefore $(1+x)^a(1-x)^b=1$, an equation to determine x.

Take the case $a=2$, $b=1$, we have $x-x^2-x^3=0$; therefore

$$x=0\quad\text{or}\quad x=\tfrac{1}{2}(-1\pm\sqrt{5}).$$

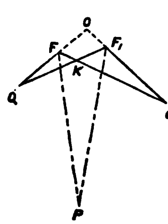
The admissible solution is $\tfrac{1}{2}(-1+\sqrt{5})$.

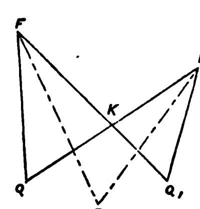
3565. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—Two tangents PQ, PQ_1 are drawn to an ellipse whose foci are F, F_1; prove that

$$\angle FQF_1+\angle F_1Q_1F=2\angle FPF_1.$$

Solution by J. J. WALKER, M.A.; A. B. EVANS, M.A.; *and others.*

This relation depends merely upon the lines PF, PF_1 bisecting the angles QFQ_1, Q_1F_1Q respectively, and amounts to this; viz., that the inclination of the bisectors of two angles is equal to half the sum of the inclinations of the sides containing one angle each to one of the sides containing the other.





Let QF, Q_1F_1 meet in O, then

$$\angle PFQ=\tfrac{1}{2}FOF_1+\tfrac{1}{2}F_1Q_1F,\quad\text{and}\quad \angle PF_1Q_1=\tfrac{1}{2}FOF_1+\tfrac{1}{2}FQF_1;$$

therefore $\angle PFQ+\angle PF_1Q_1=\angle FOF_1+\tfrac{1}{2}(\angle FQF_1+\angle F_1Q_1F)$,

but $\angle PFQ+\angle PF_1Q_1=\angle FOF_1+\angle FPF_1$;

therefore $\angle FPF_1=\tfrac{1}{2}(\angle FQF_1+\angle F_1Q_1F)$.

N.B.—If QF_1, Q_1F meet in K, $\angle FPF_1 = \frac{1}{2}(\angle FKF_1 - \angle FOF_1)$.

A property of the figure which involves all the equalities among the angles in the figure of the ellipse, is that

$$\angle Q_1FF_1 + \angle QF_1F = 2\angle FPQ \text{ or } 2\angle F_1PQ_1;$$

whether this has been noticed before or not, I am unaware.

3553. (Proposed by Professor WOLSTENHOLME.)—If P be the point, the sum of the squares of whose distances from n given straight lines is a minimum; P is the centre of gravity of the feet of the perpendiculars let fall from P on the straight line.

Solution by the Rev. R. TOWNSEND, M.A., F.R.S.; R. W. GENESE, B.A.; Rev. G. H. HOPKINS, M.A.; *and others.*

For if A, B, C, D, &c. be the feet of the several perpendiculars from P on the several lines (or planes), and A′, B′, C′, D′, &c., those for any other point P′ in the plane (or in space), then since $\Sigma(\overline{PA}^2)$ is less by hypothesis than $\Sigma(\overline{P'A'}^2)$, it is therefore less *à fortiori* than $\Sigma(\overline{P'A}^2)$, and therefore, &c.

3475. (Proposed by J. F. MOULTON, M.A.)—Find the differential and functional equations to surfaces cutting everywhere at right angles the family $z + a = xy$.

Solution by ELLEN RHODES.

Let (ξ, η, ζ) be the current coordinates of the family of surfaces required. At any point the normals to the two curves passing through it are perpendicular, or

$$\frac{d\zeta}{d\xi}\frac{dz}{dx} + \frac{d\zeta}{d\eta}\frac{dz}{dy} + 1 = 0.$$

Now from the equation to the given family we have $\dfrac{dz}{dx} = y$, $\dfrac{dz}{dy} = x$; hence the differential equation to the family is

$$\eta\frac{d\zeta}{d\xi} + \xi\frac{d\zeta}{d\eta} + 1 = 0, \text{ or } 1 + py + qx = 0.$$

Lagrange's auxiliary equations to solve this are $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{-1}$, the solution of which is $x^2 - y^2 = A$, $\log(x-y) = z + \text{const.}$, or $x - y = B\epsilon^z$.

Our solution or functional equation is therefore $x - y = \epsilon^z f(x^2 - y^2)$.

3408. (Proposed by ELIZABETH BLACKWOOD.)—A point is taken at random on a window consisting of nine equal square panes, and through this point a line is drawn in a random direction; find the respective chances of the line so drawn cutting one, two, three, four, or five panes.

I. *Solution by* HUGH MCCOLL.

I have shown in my Solution of Quest. 3440 (p. 31 of this volume), that the chance of a random line through a random point in a given rectangle cutting two given opposite sides is

$$\frac{1}{\pi}\left\{2\cot^{-1}a - a\log\frac{a^2+1}{a^2}\right\} = \phi(a) \text{ say,}$$

in which a denotes the ratio which one of the other sides has to one of the given opposite sides; and that the chance of the random line cutting two given adjacent sides is

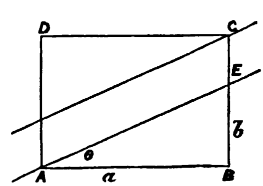
$$\tfrac{1}{4}\left\{1-\phi(a)-\phi(a^{-1})\right\} = \frac{1}{4\pi a}(a^2+1)\log(a^2+1) - \frac{a}{2\pi}\log a = \phi_1(a) \text{ say.}$$

Employing these formulæ as in my solution of Question 3440, I arrive at the following results in reference to this question:—

$p_1 = \cdot 04903 = \frac{4}{9}\phi_1(1)$;
$p_2 = \cdot 07893 = 8q$, in which $q = \frac{2}{9}\phi_1(2) - \frac{1}{9}\phi_1(1)$;
$p_3 = \cdot 32961 = 8q_1 + 4q_2 + 2\phi(3)$, in which $q_1 = \frac{1}{3}\phi_1(3) - \frac{2}{9}\phi_1(2)$,
and $q_2 = \frac{4}{9}\phi_1(1) - \frac{2}{9}\phi_1(2) - q$;
$p_4 = \cdot 37464 = 8q_3 + 4q_4$, in which $q_3 = \frac{2}{3}\phi_1(\frac{3}{2}) - 2q - q_1 - q_2 - \frac{1}{9}\phi_1(1)$,
and $q_4 = \frac{2}{3}\phi(\frac{3}{2}) - \frac{2}{3}\phi(3)$;
$p_5 = \cdot 16779 = 1 - p_1 - p_2 - p_3 - p_4$.

[The agreement between these results and those obtained experimentally, from 1150 trials, by Miss BLACKWOOD, as given in her article on p. 55 of this volume, is as close as could reasonably be expected.]

II. *Solution by* G. S. CARR.



(Fig. 1.)

If a random line be drawn through a point taken at random in the rectangle ABCD, the probability of its passing through two adjacent sides, a and b, at a given inclination θ to the side a is (see fig. 1)

$$\frac{\text{area ABE}}{\text{area ABCD}} = \frac{a^2\tan\theta}{2ab}\frac{d\theta}{\pi};$$

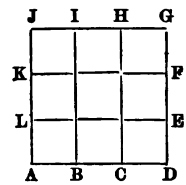
therefore the entire probability of passing through those sides is

$$\left\{\int_0^{\tan^{-1}\frac{b}{a}} \frac{a^2\tan\theta}{2}\,d\theta + \int_0^{\tan^{-1}\frac{a}{b}} \frac{b^2\tan\theta}{2}\,d\theta\right\}\frac{1}{\pi ab}$$

$$= \left\{\frac{a^2+b^2}{4}\log(a^2+b^2) - \frac{a^2}{2}\log a - \frac{b^2}{2}\log b\right\}\frac{1}{\pi ab} = \frac{\phi(a,b)}{\pi ab} \text{ say.}$$

By similar reasoning, the probability of passing through the two opposite sides b, b is $\int_0^{\tan^{-1}\frac{b}{a}} 2\,(ab - a^2 \tan\theta)\,d\theta$

$$= \left\{2ab \tan^{-1}\frac{b}{a} + 2a^2 \log a - a^2 \log(a^2+b^2)\right\}\frac{1}{\pi ab} = \frac{\psi(a,b)}{\pi ab} \text{ say.*}$$



(Fig. 2.)

Now let ADGJ (fig. 2) be a window of nine panes, the length of each $= a$, and width of each $= b$.

Let the notation p (DE, CD) be used to signify the probability of the line in question crossing the two straight lines DE and CD.

Let p_1, p_2, p_3, p_4, p_5 be the probabilities required, and first let a and b be unequal.

By reference to the figure the following deductions may be made:

I. $p\,(\text{CD, DE}) = \dfrac{\phi(a,b)}{9ab\pi}$; therefore $p_1 = \dfrac{4\phi(a,b)}{9ab\pi}$.

(We will, after this, omit writing the divisor $9ab\pi$ until the conclusion.)

II. $p\,(\text{BC, DE}) = p\,(\text{BD, DE}) - p\,(\text{CD, DE}) = \phi(2a,b) - \phi(a,b)$;

$p\,(\text{FE, CD}) = \phi(2b,a) - \phi(b,a)$, and $\phi(a,b) = \phi(b,a)$;

therefore $p_2 = 4\left\{\phi(2a,b) + \phi(2b,a) - 2\phi(a,b)\right\}$.

III. $p\,(\text{AB, DE}) = \phi(3a,b) - \phi(2a,b)$;

$p\,(\text{BC, EF}) = p\,(\text{BD, DF}) - p\,(\text{BD, DE}) - p\,(\text{CD, DF}) + p\,(\text{CD, DE})$
$= \phi(2a,2b) - \phi(2a,b) - \phi(2b,a) + \phi(a,b)$;

$p\,(\text{AL, DE}) = \psi(3a,b)$; three distinct ways only.

Thus, including the cases in which a and b are merely transposed,

$$p_3 = 4\left\{\phi(3a,b) - 2\phi(2a,b) + \phi(2a,2b) + \phi(a,b) + \phi(3b,a) - 2\phi(2b,a)\right\} + 3\left\{\psi(3a,b) + \psi(3b,a)\right\}.$$

IV. $p_4 = 4p\,(\text{AB, EF}) + 2p\,(\text{AL, EF}) + 2p\,(\text{DE, LK})$,
+ (the same with a and b transposed),

$p\,(\text{AB, EF}) = \phi(3a,2b) - \phi(3a,b) - \phi(2a,2b) + \phi(2a,b)$, as in III.;

$p\,(\text{AL, EF}) + p\,(\text{DE, LK}) = p\,(\text{AK, DF}) - p\,(\text{AL, DE}) - p\,(\text{LK, EF})$
$= \psi(3a,2b) - 2\psi(3a,b)$.

$$\text{Thus}\quad p_4 = 4\left\{\begin{array}{l}\phi(3a,2b) - \phi(3a,b) - \phi(2a,2b) + \phi(2a,b)\\ +\,\phi(3b,2a) - \phi(3b,a) - \phi(2b,2a) + \phi(2b,a)\end{array}\right\}$$
$$+\,2\left\{\psi(3a,2b) - 2\psi(3a,b) + \psi(3b,2a) - 2\psi(3b,a)\right\}.$$

V. $p_5 = 4p\,(\text{AB, FG}) + p\,(\text{AL, FG}) + p\,(\text{DE, KJ})$
$+\,p\,(\text{AB, HG}) + p\,(\text{CD, IJ})$,

$p\,(\text{AB, FG}) = \phi(3a,3b) - \phi(3a,2b) - \phi(3b,2a) + \phi(2a,2b)$, as in III.;

$p\,(\text{AL, FG}) + p\,(\text{DE, KJ}) = p\,(\text{AJ, DG}) - p\,(\text{LJ, EG}) - p\,(\text{AK, DF})$
$+\,p\,(\text{LK, EF}) = \psi(3a,3b) - 2\psi(3a,2b) + \psi(3a,b)$.

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* If we put $b = 1$, these results are identical with Mr. McColl's in the August Number of the *Educational Times*.

Thus $p_5 = 4\left\{\phi(3a, 3b) - \phi(3a, 2b) - \phi(3b, 2a) + \phi(2a, 2b)\right\}$
$+ \psi(3a, 3b) - 2\psi(3a, 2b) + \psi(3a, b) + \psi(3b, 3a) - 2\psi(3b, 2a) + \psi(3b, a)$.

Now let $a = b$, and the values of ϕ and ψ which are required will reduce to the following,—

$$\begin{aligned}
\phi(a, a) &= a^2 (\tfrac{1}{2}\log 2) \\
\phi(2a, a) &= a^2 (\tfrac{5}{4} - \tfrac{13}{4}\log 2) \\
\phi(3a, a) &= a^2 (\tfrac{5}{2} - \tfrac{9}{2}\log 3) \\
\phi(2a, 2a) &= a^2 (2\log 2) \\
\phi(3a, 2a) &= a^2 (\tfrac{13}{4}\log 13 - \tfrac{9}{2}\log 3 - 2\log 2) \\
\phi(3a, 3a) &= a^2 (\tfrac{9}{2}\log 2) \\
\psi(3a, a) &= a^2 (6\tan^{-1}\tfrac{1}{3} + 18\log 3 - 9) \\
\psi(3a, 2a) &= a^2 (12\tan^{-1}\tfrac{2}{3} + 18\log 3 - 9\log 13) \\
\psi(3a, 3a) &= a^2 (\tfrac{9}{2}\pi - 9\log 2).
\end{aligned}$$

Substituting these values in the expressions for p_1, p_2, p_3, p_4, p_5, and supplying the divisor $9ab\pi$, we obtain, when $a = b$,

$$\begin{aligned}
p_1 &= (2\log 2) \div 9\pi\mu, \\
p_2 &= (10 - 30\log 2) \div 9\pi\mu, \\
p_3 &= (62\log 2 + 72\log 3 - 54) \div 9\pi\mu + 36\tan^{-1}\tfrac{1}{3} \div 9\pi, \\
p_4 &= (62 - 58\log 2 - 72\log 3 - 10\log 13) \div 9\pi\mu + 48(\tan^{-1}\tfrac{2}{3} - \tan^{-1}\tfrac{1}{3}) \div 9\pi, \\
p_5 &= (24\log 2 - 18 + 10\log 13) \div 9\pi\mu + (12\tan^{-1}\tfrac{1}{3} - 48\tan^{-1}\tfrac{2}{3}) \div 9\pi + 1.
\end{aligned}$$

The logarithms are now taken to the base 10, and μ is the modulus. Log $9\pi\mu = 10 - 8{\cdot}9108233$. The results of the calculations are as follows,—

$p_1 = {\cdot}0490301$, $p_2 = {\cdot}0789209$, $p_3 = {\cdot}3295801$, $p_4 = {\cdot}3744796$, $p_5 = {\cdot}1679875$.

At each stage of the work the sum of the probabilities will be found to be unity, with the exception of the approximate arithmetical values, which, when added, give ·9999982. As, however, no test is thus afforded of the accuracy of the values of ϕ and ψ, I have calculated those values with more care.

With respect to Mr. McColl's Solution of Mr. Watson's Question 3440,* it appears to me that the definition therein adopted of a random line is too arbitrary. If a point be taken at random within a plane area, and a random line drawn through it, each position of the line is repeated with a frequency proportionate to the length of the intercepted part, and these *identical* situations of a straight line are counted as *different* situations. Would it not be more consistent with the notion of randomness to define a random line in a plane, as a straight line equally likely to occur in all positions? Such lines would, at each inclination, be equally distributed over the surface in parallel directions. The investigation will be precisely the same; the only alterations required will be in the values of ϕ and ψ.

In the first figure draw p and q perpendiculars from B and D upon AE. Then the probability of cutting AB, BC at an inclination θ to AB is $\dfrac{p}{p+q}\,\dfrac{d\theta}{\pi}$. Therefore the entire probability is

$$\frac{1}{\pi}\int_0^{\tan^{-1}\frac{b}{a}} \frac{a\sin\theta}{a\sin\theta + b\cos\theta}\,d\theta + \frac{1}{\pi}\int_0^{\tan^{-1}\frac{a}{b}} \frac{b\sin\theta}{b\sin\theta + a\cos\theta}\,d\theta$$

$$= \frac{ab}{\pi(a^2+b^2)}\left\{\log\frac{a^2+b^2}{4ab} + \frac{a}{b}\tan^{-1}\frac{b}{a} + \frac{b}{a}\tan^{-1}\frac{a}{b}\right\}.$$

* See pp. 31 and 66 of this volume.

The probability of cutting the two opposite sides BC, AD is

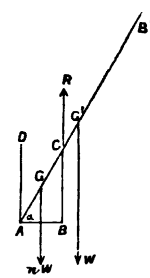
$$\frac{2}{\pi}\int^{\tan^{-1}\frac{b}{a}} \frac{b\cos\theta - a\sin\theta}{a\sin\theta + b\cos\theta} = \frac{2ab}{\pi(a^2+b^2)}\left\{\log\frac{4a^2}{a^2+b^2} + \left(\frac{b}{a}-\frac{a}{b}\right)\tan^{-1}\frac{b}{a}\right\}.$$

The practical question which has been solved in the answer to Mr. WATSON's, is the following:—

If an indefinitely small circular disc with a diameter marked upon it, be thrown against a window of four panes; what is the probability of the diameter produced cutting one, two, or three panes?

3519. (Proposed by J. J. SIDES.)—A cylinder, open at the top, stands on a horizontal plane, and a uniform rod rests partly within the cylinder, and in contact with it at its upper and lower edges. Supposing the weight of the cylinder to be n times that of the rod, r the radius of the cylinder, and α the inclination of the rod to the horizon; prove that half the length of the rod, when the cylinder is on the point of tumbling, is $(n+2)\,r\sec\alpha$.

Solution by W. SIVERLY; T. MITCHESON; *and others.*



Let ABCD represent the cylinder, AB′ the rod, G and G′ the centres of gravity of the cylinder and rod; and put $AB = 2r$, $AB' = 2x$. Since the cylinder is on the point of tumbling, the reaction R of the weights W and nW on the horizontal plane will be at B. Hence, taking moments about B, we have

$$W(x\cos\alpha - 2r) = nWr,$$

whence $$x = (n+2)\,r\sec\alpha.$$

3543. (Proposed by G. M. MINCHIN, M.A.)—Show that the first positive pedal (Bernoulli's lemniscate) and the first negative pedal of an equilateral hyperbola are mutually reciprocal polars with respect to the hyperbola.

Solution by the Rev. R. TOWNSEND, M.A., F.R.S.

To prove this pretty property, it is only necessary to show that the perpendiculars through the centre to any central radius and the corresponding tangent are conjugate diameters of the curve; which they evidently are, the radius itself and the parallel through the centre to the tangent being such in any hyperbola, and conjugate diameters continuing conjugate when turned through a right angle in an equilateral hyperbola.

NOTE ON QUESTION 2981. *By* ARTEMAS MARTIN.

The 80th set of numbers which satisfy the conditions are
10588278309438211127768625972711138460195892610538807320361440,
10588278309438211127768625972711138460195892610538807320361441,
14974086787388384990495417211933241811765094618559069827415009.

[Solutions are given on pp. 90—92 of Vol. XIV. of the *Reprint.*]

3576. (Proposed by J. J. WALKER, M.A.)—If D, E, F are the middle points of the sides BC, CA, AB of a spherical triangle ABC, and the arcs AD, BE, CF meet in O, prove that

$$\frac{\sin AD}{\sin OD} = \frac{\sin BE}{\sin OE} = \frac{\sin CF}{\sin OF} = \left\{2(\cos a + \cos b + \cos c) + 3\right\}^{\frac{1}{2}}.$$

I. *Quaternion Solution by* R. W. GENESE, B.A.

Take S the centre of the sphere for origin, and let α, β, γ be the vectors to A, B, C; then the vector of the direction AD is $\beta+\gamma$. Clearly $\alpha+\beta+\gamma$ is a vector in the plane SAD. But by symmetry it is also in the planes SBE, SCF. Therefore it is the vector of the point O in question. If

$$\alpha = x_1 i + y_1 j + z_1 k, \quad \beta = \ldots\ldots,$$

$$\begin{aligned} T(\alpha+\beta+\gamma) &= \left\{(x_1+x_2+x_3)^2 + \ldots\ldots\right\}^{\frac{1}{2}} \\ &= \left\{(x_1^2+y_1^2+z_1^2) + \ldots\ldots + 2(x_1x_2+y_1y_2+z_1z_2) + \ldots\ldots\right\}^{\frac{1}{2}} \\ &= \left\{3 + 2(\cos a + \cos b + \cos c)\right\}^{\frac{1}{2}}. \end{aligned}$$

But $$\frac{T(\alpha+\beta+\gamma)}{T\alpha} = \frac{\sin AD}{\sin OD}, \text{ and } T\alpha = 1.$$

II. *Solution by* R. TUCKER, M.A.

Let $AO = \alpha$, $OD = \alpha'$; then by two known results we have

$$2\cos(\alpha+\alpha')\cos\frac{a}{2} = \cos b + \cos c, \quad \frac{\sin\alpha}{\sin\alpha'} = 2\cos\frac{a}{2} \quad \ldots\ldots (1, 2);$$

and we want to find $\dfrac{\sin(\alpha+\alpha')}{\sin\alpha'} = \lambda$ suppose. From (1) and (2), we have

$$\cos b + \cos c = \sin\alpha \cos\alpha \cot\alpha' - \sin^2\alpha;$$

therefore $$1 + \cos b + \cos c = \frac{\cos\alpha}{\sin\alpha'}\sin(\alpha+\alpha') = \lambda\cos\alpha \ldots\ldots\ldots\ldots\ldots (3);$$

but also, from (1) and (2), $\lambda^2\sin^2\alpha = 4\cos^2\tfrac{1}{2}a - (\cos b + \cos c)^2 \ldots\ldots\ldots(4).$

$(3^2)+(4^2)$ gives $\lambda = \left\{3 + 2(\cos a + \cos b + \cos c)\right\}^{\frac{1}{2}}$, a symmetrical result.

Hence $$\frac{\sin AD}{\sin OD} = \frac{\sin BE}{\sin OE} = \frac{\sin CF}{\sin OF}.$$

3549. (Proposed by A. B. EVANS, M.A.)—To find four square numbers such that the sum of every three of them shall be a square number.

I. *Solution by* Judge SCOTT.

Let w^2, x^2, y^2, z^2 denote the numbers. Then we must have

$$w^2+x^2+y^2 = \Box \;\ldots\ldots\ldots (1), \qquad x^2+y^2+z^2 = \Box \;\ldots\ldots\ldots (2),$$
$$w^2+x^2+z^2 = \Box \;\ldots\ldots\ldots (3), \qquad w^2+y^2+z^2 = \Box \;\ldots\ldots\ldots (4).$$

Put $$w = \frac{x^2+y^2-m^2}{2m}, \text{ and } z = \frac{n^2-(x^2+y^2)}{2n};$$

then (1) and (2) are satisfied, and (3) and (4) become, after proper reduction,

$$(m^2+n^2)(x^2+y^2)^2-4m^2n^2y^2+(m^2+n^2)m^2n^2 = \Box \;\ldots\ldots\ldots (5),$$
$$(m^2+n^2)(x^2+y^2)^2-4m^2n^2x^2+(m^2+n^2)m^2n^2 = \Box \;\ldots\ldots\ldots (6).$$

Let $m^2+n^2=s^2$, and put $(5)=s^2(x^2+y^2)$; then $y=\frac{1}{2}s$, and (6) becomes

$$s^2(x^2+y^2)^2-4m^2n^2x^2+m^2n^2s^2 = \Box \;\ldots\ldots\ldots (7).$$

Assume $s(x^2-y^2)-\dfrac{2m^2n^2}{s}$ for the root of (7); then, observing that $y=\frac{1}{2}s$, we find $x=\dfrac{2m^2n^2}{s^3}$. Since $m^2+n^2=s^2$, we must take

$$m = \left(\frac{p^2-q^2}{p^2+q^2}\right)s, \text{ and } n = \left(\frac{2pq}{p^2+q^2}\right)s.$$

Hence, taking $s = 16pq(p^2-q^2)(p^2+q^2)^7$, we find

$$w = 512p^5q^5(p^2-q^2)^4+2pq(p^2+q^2)^8-8pq(p^2-q^2)^2(p^2+q^2)^6,$$
$$x = 128p^3q^3(p^2-q^2)^3(p^2+q^2)^3, \qquad y = 8pq(p^2-q^2)(p^2+q^2)^7,$$
$$z = 16p^2q^2(p^2-q^2)(p^2+q^2)^6-256p^4q^4(p^2-q^2)^5-(p^2-q^2)(p^2+q^2)^8.$$

If $p=2$ and $q=1$, then

$$w = 639604, \quad x = 3456000, \quad y = 3750000, \quad z = 832797.$$

II. *Solution by* S. BILLS.

Professor GILL, in his *Angular Analysis*, gives a masterly solution to the general problem, "To find n square numbers such that the sum of every $n-1$ of them shall be a square number." The following solution is adapted from that for the case of four square numbers:—

Let z^2, y^2, x^2, w^2 be the required squares; then we must have

$$z^2+y^2+x^2+w^2 = a^2+z^2 = b^2+y^2 = c^2+x^2 = d^2+w^2.$$

Assume
$$b = a\cos A+z\sin A, \qquad y = a\sin A-z\cos A,$$
$$c = a\cos B+z\sin B, \qquad x = a\sin B-z\cos B,$$
$$d = a\cos C+z\sin C, \qquad w = a\sin C-z\cos C;$$

then the last three conditions will be fulfilled, and it will only remain to find $y^2+x^2+w^2=a^2$, or, substituting the above values, we must have

$$a^2(\Sigma\sin^2 A-1)-az\,\Sigma\sin 2A+z^2\,\Sigma\cos^2 A = 0 \;\ldots\ldots\ldots (1).$$

Therefore $(\Sigma \sin 2A)^2 + 4\Sigma \cos^2 A - 4\Sigma \cos^2 A \,.\, \Sigma \sin^2 A = \square = k^2$,

or $k^2 = 2\Sigma(1+\cos 2A) - \Sigma(1+\cos 2A)\,\Sigma(1-\cos 2A) + (\Sigma \sin 2A)^2$

$$= -3 + 2\Sigma \cos 2A + (\Sigma \cos 2A)^2 + (\Sigma \sin 2A)^2$$

$$= 2\Sigma \cos 2A + 2\Sigma \cos(A-B) \quad\ldots\ldots\ldots\ldots(2).$$

Equation (2) may be easily reduced to the form

$$k^2 = \cos(A+B-C)\cos(A-B+C) + \cos(A+B-C)\cos(-A+B+C) + \cos(A-B+C)\cos(-A+B+C).$$

Then, if we take $A+B-C = 90°$, or $C = A+B-90°$, we shall have

$$k^2 = \sin 2A \sin 2B \quad\ldots\ldots\ldots\ldots(3),$$

which equation admits of many solutions. Dividing (3) by $4\cos^4 \frac{1}{2}B$, it becomes $\frac{1}{4}k^2 \sec^4 \frac{1}{2}B = \sin 2A \tan \frac{1}{2}B\,(1-\tan^2 \frac{1}{2}B)$;

so that, by taking $\tan \frac{1}{2}B = \sin 2A$, we have

$$\tfrac{1}{2}k \sec^2 \tfrac{1}{2}B = \sin 2A \cos 2A, \quad \text{whence } k = \frac{\sin 4A}{1+\sin 2A};$$

and from (1) we get $\dfrac{z}{a} = \dfrac{\sin 2A\,(1-2\cos 2A) + \sin^2 2A)}{1+\sin^2 2A + 2\cos 2A \sin^2 2A}$.

We may therefore take

$a = t\,(1+\sin^2 2A + 2\cos 2A \,.\, \sin^2 2A)$, $z = t \sin 2A\,(1-2\cos 2A + \sin^2 2A)$.

From which we readily find $y = t \sin A \cos 2A\,(1+2\cos 2A + \sin^2 2A)$,

$x = t \sin 2A\,(1+2\cos 2A+\sin^2 2A)$, $w = t \cos A \cos 2A\,(1-2\cos 2A + \sin^2 2A)$.

If we take $\cot \frac{1}{2}A = 2$, $t = 5^7$, we shall find

$$z = 186120, \quad y = 23828, \quad x = 102120, \quad w = 32571.$$

Again, let (3) be divided by $16\cos^4 \frac{1}{2}A \cos^4 \frac{1}{2}B$; then it gives

$$\tfrac{1}{16}k^2 \sec^4 \tfrac{1}{2}A \sec^4 \tfrac{1}{2}B = \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B\,(1-\tan^2 \tfrac{1}{2}A)(1-\tan^2 \tfrac{1}{2}B).$$

In this equation put $\tan \frac{1}{2}B = 1 - \tan \frac{1}{2}A$; then it becomes

$$\tfrac{1}{16}k^2 \sec^4 \tfrac{1}{2}A \sec^4 \tfrac{1}{2}B = \tan^2 \tfrac{1}{2}A\,(1-\tan \tfrac{1}{2}A)^2 (1+\tan \tfrac{1}{2}A)(2-\tan \tfrac{1}{2}A).$$

Now let $1+\tan \frac{1}{2}A = m \cot \frac{1}{2}\theta$, $2 - \tan \frac{1}{2}A = m \tan \frac{1}{2}\theta$,

so that $\frac{1}{16}k^2 \sec^4 \frac{1}{2}A \sec^4 \frac{1}{2}B = \tan^2 \frac{1}{2}A\,(1-\tan \frac{1}{2}A)^2 m^2$;

then $m = \frac{3}{2}\sin\theta$, $\tan \frac{1}{2}A = \frac{1}{2}(1+3\cos\theta)$, $\tan \frac{1}{2}B = \frac{1}{2}(1-3\cos\theta)$,

whence we find $k = 24 \sin\theta\,(1-9\cos^2\theta)(5+9\cos^2\theta)^2 - 36\cos\theta$.

And from (1)

$$\frac{a}{z} = \frac{3}{4}\cdot\frac{32+24\sin^2\theta\cos^2\theta - 3\sin^4\theta \mp 4\sin\theta\,(7+9\cos^2\theta)}{(1-9\cos^2\theta)(4\mp 3\sin\theta)};$$

so that we may take

$$a = 3t\left\{32+24\sin^2\theta\cos^2\theta - 3\sin^4\theta \mp 4\sin\theta\,(7+9\cos^2\theta)\right\},$$

$$z = 4t\,(1-9\cos^2\theta)(4\mp 3\sin\theta);$$

whence we find

$$y = 12t\,(1+3\cos\theta)\left\{4\sin^2 \tfrac{1}{2}\theta + 3\sin^2\theta \mp \sin\theta\,(5-3\cos\theta)\right\},$$

$$x = 12t\,(1-3\cos\theta)\left\{4\cos^2 \tfrac{1}{2}\theta + 3\sin^2\theta \mp \sin\theta\,(5+3\cos\theta)\right\}$$

$$w = 3t\sin\theta\,(1-9\cos^2\theta)(3\sin\theta \mp 4).$$

Take $\cot \frac{1}{2}\theta = 2$ and $t = \frac{5^4}{32}$; then, when the upper signs are used, we get the numbers $z=280$, $y=105$, $x=60$, $w=168$; and, by using the lower signs, $z=1120$, $y=3465$, $x=1980$, $w=672$.

[By another method of solution, Prof. GILL finds the numbers obtained by Judge SCOTT. Mr. EVANS finds the following numbers :—

1538542226307598197015, 1263916424652315771735,
1533716421893031105738, 2762941601467587299591.

3545. (Proposed by C. HARKEMA.)—The vertex of a constant angle moves round the circumference of a given circle, and one of its sides always passes through a fixed point; find the envelope of the other side.

I. *Solution by the* REV. W. H. LAVERTY, M.A.

Reciprocating, we get this problem :—

T being a tangent to a conic whose focus is S, and L being a line through S making a constant angle with T and meeting it in P, find the locus of P.

Now the locus of P, in

an ellipse, is a circle having S inside it;
a hyperbola, „ „ outside it;
a parabola, is a tangent and the line at infinity;

therefore in the original problem, if the fixed point Q

is inside the circle, the envelope is an ellipse, focus Q;
is outside „ „ a hyperbola, „
is on „ „ two points on the circle.

II. *Solution by the* REV. DR. BOOTH, F.R.S.

Let a be the radius of the circle, and c the distance of the point from the centre. Taking this point for the origin, the equation of the circle is

$$(x-c)^2+y^2 = a^2 \quad \text{.................................(1.)}$$

Let m be the tangent of the given angle, and ϕ, ϕ_1 the angles which the two lines make with the axis of x; then $\tan\phi = \frac{y}{x}$, $\tan\phi_1 = \frac{\xi}{v}$, ξ and v being the tangential coordinates of the enveloping lines. Hence

$$m = \frac{\xi x + yv}{xv - y\xi} = \frac{1}{xv-y\xi} \text{ ....(2),} \quad \text{since } x\xi + yv = 1 \text{ ....(3).}$$

From (2) and (3) we have $x = \frac{v+m\xi}{m(\xi^2+v^2)}$, $y = \frac{mv-\xi}{m(\xi^2+v^2)}$.

Substituting these values in (1), and putting for m its value $\tan\theta$, we get for the tangential equation of the envelope

$$(a^2-c^2)\sin^2\theta\,(\xi^2+v^2) + 2c\sin\theta\,(v\cos\theta+\xi\sin\theta) = 1 \text{ ....(4).}$$

Comparing this with the normal tangential equation of a conic, viz.,

$$\alpha\xi^2+\alpha_1 v^2+2\beta\xi v+2\gamma\xi+2\gamma_1 v = 1 \text{ ....(5),}$$

since $\alpha = \alpha_1$ and $\beta = 0$, the focus is at the origin. The axes are found by the formula

$$2A^2 = (\alpha+\gamma^2)+(\alpha_1+\gamma_1^2)\pm\sqrt{\left\{[(\alpha+\gamma^2)-(\alpha_1+\gamma_1^2)]^2+4(\beta+\gamma\gamma_1)^2\right\}}.$$

Now if we substitute for $\alpha, \alpha_1, \beta, \gamma, \gamma_1$, the values given by (4), we shall get $$A^2 = a^2\sin^2\theta,\quad B^2 = (a^2-c^2)\sin^2\theta.$$

Hence $e = \dfrac{c}{a}$, or the eccentricity is independent of the angle θ; hence all the loci, whatever be the angle, are uni-confocal and similar sections.

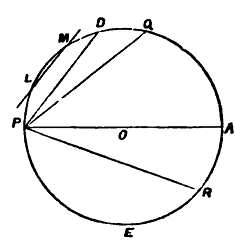
Let D be the distance of the centre from the origin; then

$$D^2 = \gamma^2+\gamma_1^2 = c^2\sin^2\theta.$$

The coordinates of the centre of the curve are $\gamma = c\sin^2\theta$, and $\gamma_1 = c\sin\theta\cos\theta$; the distance of the centre of the curve from the origin is $D = c\sin\theta$. When $c=a$, $e=1$, and the curve becomes an infinitesimal parabola, of which one vertex is on the circle at the origin, the other at the extremity of the chord of the segment which contains the given angle θ, and the coordinates of the centre of this infinitesimal parabola are $a\sin^2\theta$ and $a\sin\theta\cos\theta$.

3290. (Proposed by S. Watson.)—Through any point in the circumference of a given circle, two lines are drawn at random, and a third line is drawn in a random direction, but so as to cut the circle. Find the respective chances of the last named line intersecting neither, one, or both of the former lines, within the circle.

Solution by the Proposer.



Let O be the centre of the given circle, PA a diameter, PQ, PR any two lines through P, LM any random line cutting the circle in L, M; and draw PD parallel to LM. Put PO = 1, $\angle$ APQ = θ, APR = θ_1, and APD = ϕ. Then the distance of PD from O is $\sin\phi$, and the number of lines that can be drawn parallel to PD to cut the circle is $1-\sin\phi$. Hence the number of lines that can be drawn to cut the arc PDQ in all possible ways, is

$$2\int_\theta^{\frac{1}{2}\pi}(1-\sin\phi)\,d\phi = \pi-2\theta-2\cos\theta \ldots (1).$$

Similarly for the arcs PER, QAR the numbers are

$$\pi-2\theta_1-2\cos\theta_1, \text{ and } 2(\theta+\theta_1)-2\sin(\theta+\theta_1)\ldots\ldots\ldots\ldots (2,3).$$

Hence doubling because PQ and PR may be interchanged; the number of ways the line LM can be drawn to cut neither PR nor PQ, when PQ and PR are drawn in all directions, is

$$2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}d\theta\int_{-\theta}^{\frac{1}{2}\pi}d\theta_1\left\{(1)+(2)+(3)\right\} = 2\pi(\pi^2-6)\ldots\ldots\ldots\ldots (4).$$

Again, the number of ways the line LM can cut the arc PQR is

$$\pi + 2\theta_1 - 2\cos\theta_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

and therefore the number of ways it can cut the line PQ is

$$(5)-(1)-(3) = 2\sin(\theta+\theta_1)+2\cos\theta-2\cos\theta_1 \ldots\ldots\ldots\ldots (6).$$

Similarly the number of ways the line PR can be cut is

$$2\sin(\theta+\theta_1)-2\cos\theta+2\cos\theta_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Hence, doubling as before, the number of ways either PQ or PR can be cut by LM for all positions of PQ, PR, is

$$2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{-\theta}^{\frac{1}{2}\pi} d\theta_1\, 4\sin(\theta+\theta_1) = 8\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Lastly, the number of ways the line LM can be drawn to cut the circle is 2π; and therefore the number cutting both PQ and PR is

$$2\pi-(1)-(2)-(3)-(6)-(7) = 2\cos\theta+2\cos\theta_1-2\sin(\theta+\theta_1) \ldots\ldots (9).$$

Hence the total number of ways of cutting both PQ and PR is

$$2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta \int_{-\theta}^{\frac{1}{2}\pi} d\theta_1\, (9) = 4\pi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Dividing (4), (8), (10) by $2\pi^3$, the total number of ways the lines PQ, PR, LM can be drawn, we have the chances required; viz.,

$$1-\frac{6}{\pi^2},\ \frac{4}{\pi^2},\ \text{and}\ \frac{2}{\pi^2}\ \text{respectively.}$$

3486. (Proposed by H. McColl.)—From any point C in space a straight line is drawn in a random direction, meeting the surface of a given solid (or the surfaces of various given solids) at the variable point P. Show that the average volume of the sphere of which CP is the radius is equal to the volume of the given solid (or given solids); the sphere to be reckoned negative when P is an entrance point, positive when P is a point of exit, and zero when the random line misses all the solids and the point P is imaginary. [See Question 3370.]

Solution by G. S. Carr.

Let α = an element of surface of a concentric sphere of radius unity, and let CP = r. Then the average defined, taken only over the area α, for all positions of α within a conical surface circumscribing the given solid, will be $\frac{4}{3}\pi r^3 . \frac{\alpha}{4\pi} = \frac{r^3\alpha}{3}$; and the sum of such expressions, for all values of r within the same limits, will be the average required. But $\frac{1}{3}r^3\alpha$ = volume of a small conical element of height r; and the sum of all such solid elements within the aforesaid limits, considering r negative when it does not pass through the given solid, is evidently the volume of the solid.

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17